ADVANCED MATHEMATICAL MODELS FOR THE DESIGN AND OPTIMIZATION OF LOW-INTERFERENCE WIRELESS MULTICARRIER SYSTEMS

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Abstract—Because of their numerous advantages, multicarrier (MC) modulation techniques such as orthogonal frequency division multiplexing (OFDM) are very attractive for broadband wireless communications. Here, we consider the application of frame theory and related mathematical models to pulse-shaping MC systems transmitting over doubly dispersive fading channels (i.e., random time-varying channels with memory). An analysis of the intersymbol and intercarrier interference occurring in such systems demonstrates that the time and frequency concentration of the transmit and receive pulses is of central importance for low interference. We prove the existence of such jointly concentrated pulse pairs by adapting recent mathematical results on Weyl-Heisenberg frames to the MC context. Furthermore, we propose pulse optimization procedures that minimize interference by exploiting the design freedom existing for redundant MC systems. Numerical results demonstrate that our optimized pulse-shaping MC systems can outperform conventional cyclic-prefix OFDM systems.

1. Introduction

For broadband wireless communications, multicarrier (MC) modulation techniques are attractive due to their numerous desirable properties. In particular, an MC technique known as *cyclic-prefix orthogonal frequency division multiplexing* (CP-OFDM) [2–5] is currently used in several wireless communication standards. CP-OFDM uses rectangular transmit and receive pulses whose frequency concentration is rather poor, however. More advanced MC techniques such as *pulse-shaping OFDM* and *biorthogonal frequency division multiplexing* (BFDM) [1, 2, 6–16], though rarely used up to now in practical systems, have several advantages over traditional CP-OFDM. One of these advantages, reduced intersymbol/intercarrier interference (ISI/ICI) [1, 7], will be an important requirement for future mobile wireless systems that will have to tolerate faster channel variations, i.e., larger Doppler frequency shifts due to higher carrier frequencies and higher mobile velocities.

For such advanced MC systems, the design of the OFDM/BFDM transmit and receive pulses has been considered in [7, 9–15, 17, 18]. In particular, in [11, 14, 15] the duality of MC systems and Weyl-Heisenberg (or Gabor) frames is applied to the design of OFDM and BFDM systems. In a similar spirit, [7] proposes an optimization procedure for BFDM systems that builds on frames and Riesz sequences and explicitly accounts for channel effects. In [12], hexagonal time-frequency lattices are advocated, and sphere-packing arguments are used to construct a pulse through orthogonalization of a Gaussian function. [18] investigates the construction of pulses that are well-concentrated in time and frequency separately.

In this paper, motivated by the approach of [7, 11, 12], we apply frame theory, Riesz sequences, Gabor systems, and associated mathematical results to pulse-shaping MC systems transmitting over doubly dispersive fading channels. We present the following results.

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- We show that there exist (bi)orthogonal transmit and receive pulses that *both* are well concentrated with respect to time and frequency. More specifically, we use a recent, deep mathematical result from [19] to provide explicit expressions for the achievable time and frequency decay of the transmit/receive pulses in terms of their cross-ambiguity functions. This is important because, as we will show, good time-frequency concentration is essential for small ISI/ICI, and because in practice finite-length pulses have to be used.
- We present two different pulse optimization procedures that exploit the design freedom existing for redundant MC systems to explicitly minimize the ISI/ICI.
- We compare the performance of our optimized pulse-shaping OFDM/BFDM systems with that of standard CP-OFDM systems (cf. also [12]). It is shown that pulse-shaping MC systems can outperform CP-OFDM systems for realistic spectral efficiencies or, equivalently, redundancies. (We note that most previous numerical results pertained to spectral efficiencies ≤ 0.5, which are rarely used in practice.)

This paper is organized as follows. The MC system model is reviewed in Section 2. In Section 3, we present a frame-theoretic computation of biorthogonal pulse pairs. In Section 4, we prove the existence of biorthogonal pulses with good time-frequency concentration. Two methods for designing pulses with minimum ISI/ICI are described in Section 5. Finally, the performance of optimized pulse-shaping MC systems is assessed in Section 6.

2. MC System Model

In this section, we review a compact mathematical formulation of a generic MC system including a random time-varying channel [20, 21]. This is a unifying formulation for CP-OFDM, pulse-shaping OFDM, and BFDM systems.

2.1. **MC Modulator and Demodulator.** For an MC system with K subcarriers, symbol period T, and subcarrier frequency spacing F, the equivalent baseband transmit signal is given by

$$s(t) \,=\, \sum_{l=-\infty}^{\infty} \sum_{k=0}^{K-1} a_{l,k}\, g_{l,k}(t)\,. \tag{1}$$

Here, $a_{l,k}$ denotes the data symbol at symbol time $l \in \mathbb{Z}$ and subcarrier $k \in \{0,\ldots,K-1\}$, and $g_{l,k}(t)$ is a time-frequency (TF) shifted version of an elementary *transmit pulse* $g(t)$:

$$g_{l,k}(t) \triangleq g(t-lT)\, e^{j2\pi kF(t-lT)}. \tag{2}$$

The set of functions $\{g_{l,k}(t)\}$, $(l,k) \in \mathbb{Z}^2$ is known as a *Weyl-Heisenberg (WH) function set*.

At the receiver (demodulator), the inner products of the received signal $r(t)$ with TF shifted versions $\gamma_{l,k}(t) \triangleq \gamma(t-lT)\, e^{j2\pi kF(t-lT)}$ of an elementary *receive pulse* $\gamma(t)$ are computed:

$$x_{l,k} \,\triangleq\, \langle r, \gamma_{l,k}\rangle \,=\, \int_t r(t)\, \gamma_{l,k}^*(t)\, dt\,. \tag{3}$$

(Integrals are from $-\infty$ to ∞.) In the theoretical case of an ideal channel, we have $r(t) = s(t)$. Here, the demodulated symbols $x_{l,k}$ equal the transmit symbols $a_{l,k}$ iff the transmit pulse $g(t)$ and the receive pulse $\gamma(t)$ satisfy the *biorthogonality property*

$$\langle g, \gamma_{l,k}\rangle \,=\, \delta_l\, \delta_k\,. \tag{4}$$

For OFDM, in particular, we have $\gamma(t) = g(t)$ and thus (4) reduces to the orthogonality property $\langle g, g_{l,k}\rangle = \delta_l\, \delta_k$.

A necessary condition for the biorthogonality property (4) is $TF \geq 1$ [22]. However, TF is typically chosen only slightly larger than 1 because the spectral efficiency (i.e., the data rate normalized by the transmit bandwidth) is proportional to $1/(TF)$. Practical CP-OFDM systems use $TF = 1.03 \ldots 1.25$. On the other hand, larger values of TF increase the freedom in designing pulses $g(t)$ and $\gamma(t)$ satisfying (4).

2.2. **Channel.** The wireless channel is modeled as a random linear time-varying system $\mathbb{H}$ with time-varying impulse response $h(t,\tau)$ [20, 21]. The channel output (received signal) is thus given by

$$r(t) \,=\, (\mathbb{H}\, s)(t) \,=\, \int_\tau h(t,\tau)\, s(t-\tau)\, d\tau\,. \tag{5}$$

(We do not include an additive noise component because we are not interested in noise effects in this paper.) The statistical description of the channel is greatly simplified by the *wide-sense stationary uncorrelated scattering* (WSSUS) assumption [20, 21]

$$\mathrm{E}\,\{h(t,\tau)h^*(t',\tau')\} = R_{\mathbb{H}}(t-t',\tau)\, \delta(\tau-\tau'),$$

where $\mathrm{E}\{\cdot\}$ is the expectation operator and $R_{\mathbb{H}}(\Delta t, \tau)$, the channel's *time-delay correlation function*, describes the second-order statistics of $\mathbb{H}$. Alternatively, the *scattering function* defined as [20, 21]

$$C_{\mathbb{H}}(\tau, \nu) = \int_{\Delta t} R_{\mathbb{H}}(\Delta t, \tau)\, e^{-j2\pi\nu\Delta t}\, d\Delta t\,,$$

describes the second-order statistics of $\mathbb{H}$ in terms of time delay τ and Doppler frequency shift ν. The integral of $C_{\mathbb{H}}(\tau, \nu)$ over the delay-Doppler plane, $\sigma_{\mathbb{H}}^2 = \int_\tau \int_\nu C_{\mathbb{H}}(\tau, \nu)\, d\tau\, d\nu$, is known as the channel's *path gain*. The scattering function of wireless channels is effectively supported within a rectangular region $\mathcal{R} \triangleq [0, \tau_{\max}] \times [-\nu_{\max}, \nu_{\max}]$ of area $2\tau_{\max}\nu_{\max} \ll 1$ [7, 21, 23].

2.3. **Matrix Representation and Almost Diagonalization.** The modulator (1), channel (5), and demodulator (3) can be combined to obtain the following linear relation connecting the transmit symbols $a_{l,k}$ and the demodulated symbols $x_{l,k}$ [7]:

$$x_{l,k} = \sum_{l'=-\infty}^{\infty} \sum_{k'=0}^{K-1} H_{l,k;l',k'}\, a_{l',k'}\,, \qquad \text{with} \quad H_{l,k;l',k'} \triangleq \langle \mathbb{H}\, g_{l',k'}, \gamma_{l,k} \rangle\,. \tag{6}$$

In this sum, the terms with $(l', k') \neq (l, k)$ describe the "crosstalk" between different symbols (ISI and ICI). The parameters of MC systems are usually chosen such that these ISI/ICI terms are negligible and thus (6) simplifies to an "almost diagonal" relation:

$$x_{l,k} \approx h_{l,k}\, a_{l,k}\,, \qquad \text{with} \quad h_{l,k} \triangleq H_{l,k;l,k}\,. \tag{7}$$

This greatly simplifies the receiver because a simple scalar equalization can be used. It follows from (7) that the transmit pulses $g_{l,k}(t)$ must be approximate eigenfunctions of the channel, i.e., $(\mathbb{H}g_{l,k})(t) \approx h_{l,k}\, g_{l,k}(t)$ [23, 24] (together with (4), this implies (7)).

As will be shown in Section 5.1, the approximate input-output relation (7) requires the biorthogonal pulses $g(t)$ and $\gamma(t)$ to be well TF concentrated. In Section 4, we will show that such pulses indeed exist.

3. Frame-Theoretic Computation of MC Pulses

We now commence our discussion of the construction of transmit pulses $g(t)$ and receive pulses $\gamma(t)$ with desirable properties. First we show how the theory of WH (Gabor) frames can be applied to the computation of a receive pulse $\gamma(t)$ satisfying the biorthogonality relation (4) for a given transmit pulse $g(t)$ [11, 15].

3.1. **Some Mathematical Fundamentals.** We first review some mathematical fundamentals concerning WH Riesz sequences, WH frames, and a duality relation connecting them. Our subsequent development applies to an infinite number of subcarriers ($K = \infty$); the results to be obtained can be expected to be meaningful if the actual number of subcarriers is reasonably large.

Weyl-Heisenberg Riesz Sequences. Evidently, a necessary condition for successful data transmission is the possibility to recover the transmit symbols $a_{l,k}$ from the transmit signal $s(t)$. Mathematically, this amounts to the requirement that the WH function set $\{g_{l,k}(t)\}$, $(l, k) \in \mathbb{Z}^2$ in (2) constitutes a *Riesz sequence* [22], i.e., there exist constants $A', B' > 0$ such that

$$A'\|a\|_{\ell^2}^2 \leq \Big\| \sum_{l,k} a_{l,k}\, g_{l,k} \Big\|_{L^2}^2 \leq B'\|a\|_{\ell^2}^2\,, \qquad \text{for all } a_{l,k} \in \ell^2(\mathbb{Z}^2)\,, \tag{8}$$

with $\|a\|_{\ell^2}^2 = \sum_{l,k} |a_{l,k}|^2$ and $\|s\|_{L^2}^2 = \int_t |s(t)|^2\, dt$. It can be shown [22] that a necessary condition for a WH Riesz sequence is that the TF lattice constants T, F satisfy $TF \geq 1$. If $\{g_{l,k}(t)\}$ is a WH Riesz sequence, there exists a (nonunique) WH Riesz sequence $\{\gamma_{l,k}(t)\}$ that satisfies the biorthogonality condition (4). Thus, the transmit symbols $a_{l,k}$ can be recovered according to (3) (assuming $r(t) = s(t)$).

If $\langle g, g_{l,k} \rangle = \delta_l\, \delta_k$ holds in addition to (8), then the Riesz sequence $\{g_{l,k}(t)\}$ is an *orthogonal* sequence that corresponds to a pulse-shaping OFDM (and not just BFDM) system.

Weyl-Heisenberg (Gabor) Frames. A WH function set $\{\widetilde{g_{l,k}}(t) = g(t - l\tilde{T})\, e^{j2\pi k\tilde{F}t}\}$, $(l, k) \in \mathbb{Z}^2$ is called a *WH frame* [22] if there exist constants $A, B > 0$ (known as *frame bounds*) such that

$$A\|x\|_{L_2}^2 \leq \sum_{l,k} |\langle x, \widetilde{g_{l,k}} \rangle|^2 \leq B\|x\|_{L_2}^2\,, \qquad \text{for all } x(t) \in L_2(\mathbb{R})\,.$$

If $\{\widetilde{g_{l,k}}(t)\}$ is a frame, any $x(t) \in L_2(\mathbb{R})$ can be stably recovered from the numbers $\langle x, \widetilde{g_{l,k}} \rangle$ by means of the *frame expansion* $x(t) = \sum_{l,k} \langle x, \widetilde{g_{l,k}} \rangle\, \widetilde{\gamma_{l,k}}(t)$. Here, $\widetilde{\gamma_{l,k}}(t)$ is a *dual WH frame* that is based on a (generally nonunique) *dual pulse* $\gamma(t)$. A necessary condition for a WH frame is that the TF lattice constants $\tilde{T}, \tilde{F}$ satisfy $\tilde{T}\tilde{F} \leq 1$ [22]. A frame $\{\widetilde{g_{l,k}}(t)\}$ is said to be *tight* if the frame bounds A, B are equal. For a tight frame, the dual pulse is a multiple of the original pulse: $\gamma(t) = \frac{\tilde{T}\tilde{F}}{A}\, g(t)$.

Ron-Shen Duality. A fundamental relation of WH frames and WH Riesz sequences is formulated by the *Ron-Shen duality theorem* [25, 26]. Let $\{g_{l,k}(t)\}$ and $\{\gamma_{l,k}(t)\}$ be WH sets on the lattice (T, F). Furthermore let $\{\widetilde{g_{l,k}}(t)\}$ and $\{\widetilde{\gamma_{l,k}}(t)\}$ be the corresponding WH sets (using the same $g(t)$ and $\gamma(t)$) on the *adjoint lattice* $(\tilde{T}, \tilde{F}) = (1/F, 1/T)$. Then

- $\{g_{l,k}(t)\}$ is a Riesz sequence iff $\{\widetilde{g_{l,k}}(t)\}$ is a frame;
- $\{g_{l,k}(t)\}$ and $\{\gamma_{l,k}(t)\}$ are biorthogonal Riesz sequences iff $\{\widetilde{g_{l,k}}(t)\}$ and $\{\widetilde{\gamma_{l,k}}(t)\}$ are dual frames;
- $\{g_{l,k}(t)\}$ is an orthogonal Riesz sequence iff $\{\widetilde{g_{l,k}}(t)\}$ is a tight frame.

3.2. **Application to Pulse Computation.** The Ron-Shen duality relation allows us to reduce the computation of pulses inducing (bi)orthogonal WH Riesz sequences to the computation of *dual WH frames on the adjoint lattice*, for which frame-theoretic methods are available [11, 15, 22, 27]. First, we choose lattice constants T, F and a pulse $g(t)$ such that the WH set $\{\widetilde{g_{l,k}}(t)\}$ with $\widetilde{g_{l,k}}(t) = g(t - l\tilde{T})\, e^{j2\pi k\tilde{F}t}$ on the adjoint lattice $(\tilde{T}, \tilde{F}) = (1/F, 1/T)$ is a WH frame (this requires $TF \geq 1$). For the further procedure, two cases have to be distinguished:

- For a BFDM system, the transmit pulse is directly given by $g(t)$, and the biorthogonal receive pulse with minimum L^2-norm is given by the *canonical dual pulse* $\gamma_c(t) = TF\, (\tilde{\mathbb{S}}^{-1}g)(t)$, where the invertible *frame operator* $\tilde{\mathbb{S}}$ is defined by
$$(\tilde{\mathbb{S}}x)(t) = \sum_{l,k} \langle x, \widetilde{g_{l,k}} \rangle\, \widetilde{g_{l,k}}(t)\,.$$
- For an OFDM system, the transmit pulse and the receive pulse are both equal to $g^{\perp}(t) = \sqrt{TF}\, (\tilde{\mathbb{S}}^{-1/2}g)(t)$.

An efficient numerical implementation of this computation procedure can be based on conjugate gradient and matrix factorization methods [27] as well as Zak transform methods [11, 15]. Alternatively, $\gamma_c(t)$ can be computed by inserting the expansion $\gamma_c(t) = \sum_{l,k} c_{l,k}\, g_{l,k}(t)$ into the biorthogonality condition (4), whereby a linear equation in the unknown coefficients $c_{l,k}$ is obtained [27].

For $TF > 1$, the biorthogonal receive pulse corresponding to a given transmit pulse $g(t)$ is not unique. In fact, all biorthogonal pulses are of the form $\gamma(t) = \gamma_c(t) + \psi(t)$ where $\gamma_c(t) = TF\, (\tilde{\mathbb{S}}^{-1}g)(t)$ is the canonical biorthogonal pulse and $\psi(t)$ is an arbitrary function in the orthogonal complement of the space $\mathcal{G} \triangleq \text{span}\{g_{l,k}(t)\}$ [22]. The resulting freedom in designing a biorthogonal receive pulse for a given transmit pulse can be exploited for pulse optimization, as discussed in Section 5.

4. Time-Frequency Concentration of MC Pulses

We have shown in the previous section how the canonical biorthogonal pulse $\gamma_c(t)$ and the orthogonalized pulse $g^{\perp}(t)$ can be constructed for a given transmit pulse $g(t)$. In the MC context, pulses with good TF concentration are desirable because, as we will see in Section 5.1, the ISI/ICI in the case of doubly dispersive channels is determined by the joint TF concentration of $g(t)$ and $\gamma(t)$. More specifically, the ISI/ICI power depends on the cross-ambiguity function (CAF) of $\gamma(t)$ and $g(t)$ that is defined as [22]

$$A_{\gamma,g}(\tau, \nu) \triangleq \int_t \gamma(t)\, g^*(t - \tau)\, e^{-j2\pi\nu t} dt\,.$$

We will now use a recent mathematical result [19] to show that the canonical dual $\gamma_c(t)$ and the orthogonalized pulse $g^{\perp}(t)$ as defined in Section 3.2 inherit the TF localization properties of $g(t)$. Let us say that a pulse $g(t)$ is *polynomially localized* of degree $s \geq 0$ if

$$\int_\tau \int_\nu |A_{g,g}(\tau, \nu)|\, w(\tau, \nu)\, d\tau\, d\nu \;<\; \infty\,, \tag{9}$$

with weight function $w(\tau, \nu) \triangleq (1 + |\tau/T_0| + |\nu T_0|)^s$, where $T_0 > 0$ is an arbitrary normalization time constant. The polynomial localization condition implies that $|A_{g,g}(\tau, \nu)| \leq c/w(\tau, \nu)$ for some $c > 0$ and that all moments of $A_{g,g}(\tau, \nu)$ up to order s are finite, i.e., $\int_\tau \int_\nu |A_{g,g}(\tau, \nu)|\, \big|\tau/T_0\big|^m |\nu T_0|^n\, d\tau\, d\nu < \infty$ for $m + n \leq s$. A stronger measure of TF localization is obtained by using in (9) the sub-exponential weight function $w(\tau, \nu) = \exp\big(b(|\tau/T_0| + |\nu T_0|)^\beta\big)$ with $b > 0$ and $0 < \beta < 1$. We thus say that $g(t)$ is *sub-exponentially localized* if (9) holds with a sub-exponential weight $w(\tau, \nu)$. The sub-exponential localization condition implies that $|A_{g,g}(\tau, \nu)| \leq c\, e^{-b(|\tau/T_0| + |\nu T_0|)^\beta}$ for some $c > 0$ and that *all* moments of $A_{g,g}(\tau, \nu)$ are finite.

The next result asserts the existence of transmit and receive pulses that *simultaneously* have excellent TF concentration properties.

Localization Theorem. Let $\{\widetilde{g_{l,k}}(t) = g(t-l/F)\, e^{j2\pi kt/T}\}$, $(l,k) \in \mathbb{Z}^2$ with $TF \geq 1$ be a WH frame. Then, if $g(t)$ is polynomially localized of degree s (sub-exponentially localized), the canonical biorthogonal pulse $\gamma_c(t) = TF\,(\tilde{\mathbb{S}}^{-1}g)(t)$ and the orthogonalized pulse $g^{\perp}(t) = \sqrt{TF}\,(\tilde{\mathbb{S}}^{-1/2}g)(t)$ are also polynomially localized of degree s (sub-exponentially localized), and furthermore the CAF of $\gamma_c(t), g(t)$ satisfies

$$\int_\tau \int_\nu |A_{\gamma_c,g}(\tau,\nu)|\, w(\tau,\nu)\, d\tau\, d\nu \;<\; \infty\,,$$

where $w(\tau,\nu)$ is the appropriate polynomial (sub-exponential) weight function.

The proof of this theorem is an application of the results derived in [19] in combination with Ron-Shen duality theory. The theorem agrees well with previous empirical observations [11].

5. Optimization of MC Pulses

In this section, we present methods for an optimization of the transmit and receive pulses. The optimality criterion is minimum ISI/ICI under the assumption of a WSSUS channel with given second-order statistics (scattering function $C_{\mathbb{H}}(\tau,\nu)$, cf. Section 2.2). We first present an expression for the mean ISI/ICI power which shows that for low ISI/ICI, an MC system should use pulses that are jointly well localized in time and frequency.

5.1. **Mean ISI/ICI Power.** The *mean ISI/ICI power* is defined as the mean-square error of the approximation (7), i.e., $\sigma_{\mathrm{I}}^2 \triangleq \mathrm{E}\{|x_{l,k} - h_{l,k}\,a_{l,k}|^2\}$, whereas the mean power of the desired component $h_{l,k}\,a_{l,k}$ is defined as $\sigma_{\mathrm{D}}^2 \triangleq \mathrm{E}\{|h_{l,k}\,a_{l,k}|^2\}$. Note that σ_{I}^2 and σ_{D}^2 do not depend on l and k since the channel is assumed WSSUS. In what follows, we consider a pulse-shaping MC system with an infinite number of subcarriers. We also assume that the (random) data symbols $a_{l,k}$ are independent and identically distributed (i.i.d.) with zero mean and mean power $\mathrm{E}\{|a_{l,k}|^2\} = \sigma_a^2$. One can then show (cf. [7] and, for the special case of a CP-OFDM system, also [28]) the following expression for the mean ISI/ICI power:

$$\sigma_{\mathrm{I}}^2 \;=\; \sigma_a^2 \int_\tau \int_\nu Q_{\mathbb{H}}(\tau,\nu)\, |A_{\gamma,g}(\tau,\nu)|^2\, d\tau\, d\nu\,. \tag{10}$$

Here,

$$Q_{\mathbb{H}}(\tau,\nu) \;\triangleq \sum_{(l,k)\neq(0,0)} C_{\mathbb{H}}(\tau - lT, \nu - kF)$$

is a periodized version of the scattering function $C_{\mathbb{H}}(\tau,\nu)$ with the term at the origin ($(l,k) = (0,0)$) suppressed. Similarly, the mean power of the desired component can be expressed as

$$\sigma_{\mathrm{D}}^2 \;=\; \sigma_a^2 \int_\tau \int_\nu C_{\mathbb{H}}(\tau,\nu)\, |A_{\gamma,g}(\tau,\nu)|^2\, d\tau\, d\nu\,. \tag{11}$$

According to (10), σ_{I}^2 will be small if $A_{\gamma,g}(\tau,\nu)$ has small overlap with $Q_{\mathbb{H}}(\tau,\nu)$. Such a behavior is favored by a good TF concentration of both $g(t)$ and $\gamma(t)$, by a small support region $\mathcal{R}$ of $C_{\mathbb{H}}(\tau,\nu)$ (i.e., the channel is only weakly dispersive), and by large lattice constants T and F. Unfortunately, the channel dispersion is beyond the control of the designer and large T and F entails poor spectral efficiency. Therefore, it remains to design pulses $g(t)$ and $\gamma(t)$ that are *jointly* well TF concentrated in the sense of quick decay of their CAF $A_{\gamma,g}(\tau,\nu)$. Our localization theorem in Section 4 proves the existence of such well-localized pulse pairs: for a given transmit pulse $g(t)$ with polynomial (sub-exponential) TF localization, the *canonical* biorthogonal receive pulse $\gamma_c(t)$ also has polynomial (sub-exponential) TF localization and $A_{\gamma_c,g}(\tau,\nu)$ exhibits fast decay. However, for $TF > 1$ there are an infinite number of other biorthogonal receive pulses besides the canonical one. Therefore, we next propose a pulse optimization procedure that exploits this design freedom to improve on the canonical biorthogonal receive pulse in terms of ISI/ICI.

5.2. **Optimization of Biorthogonal Receive Pulse.** We assume that we are given a transmit pulse $g^{(0)}(t)$ and calculate the receive pulse $\gamma(t)$ minimizing the ISI/ICI power σ_{I}^2 subject to the biorthogonality condition (4). Any receive pulse biorthogonal to $g^{(0)}(t)$ can be written as [22]

$$\gamma(t) \;=\; \gamma_c(t) + \psi(t)\,, \tag{12}$$

where $\gamma_c(t)$ is the *canonical* biorthogonal pulse and $\psi(t)$ is an arbitrary function in $\mathcal{G}^{\perp}$, the orthogonal complement space of $\mathcal{G} = \mathrm{span}\{g_{l,k}^{(0)}(t)\}$. We can express $\psi(t)$ in terms of an orthonormal basis $\{u_i(t)\}$ of $\mathcal{G}^{\perp}$ as $\psi(t) =$

$\sum_i c_i\, u_i(t)$, with $c_i = \langle \psi, u_i \rangle$. Using (12), the minimization of σ_{I}^2 with respect to $\gamma(t)$ under the biorthogonality constraint (4) can be formulated as an *unconstrained* minimization with respect to the expansion coefficients c_i.

Inserting (12) into (10), we obtain an expression of σ_{I}^2 in terms of the coefficients c_i. It is then straightforward to show that the c_i's minimizing σ_{I}^2 are the solution to the linear equation

$$\mathbf{B}\mathbf{c} = -\mathbf{b}\,,$$

where $[\mathbf{c}]_i = c_i$ and

$$[\mathbf{B}]_{i,j} \,=\, \int_\tau \int_\nu Q_{\mathbb{H}}(\tau,\nu)\, A^*_{u_i,g^{(0)}}(\tau,\nu)\, A_{u_j,g^{(0)}}(\tau,\nu)\, d\tau\, d\nu\,,$$

$$[\mathbf{b}]_i \,=\, \int_\tau \int_\nu Q_{\mathbb{H}}(\tau,\nu)\, A^*_{u_i,g^{(0)}}(\tau,\nu)\, A_{\gamma^{(0)},g^{(0)}}(\tau,\nu)\, d\tau\, d\nu\,.$$

Therefore, the biorthogonal receive pulse minimizing the ISI/ICI power σ_{I}^2 is obtained as

$$\gamma^{\mathrm{opt}}(t) \,=\, \gamma_c(t) + \sum_i c_i^{\mathrm{opt}}\, u_i(t) \qquad \text{with } \mathbf{c}^{\mathrm{opt}} \,=\, -\mathbf{B}^{-1}\mathbf{b}\,. \tag{13}$$

In practical digital implementations, the matrix $\mathbf{B}$ and the vectors $\mathbf{b}$ and $\mathbf{c}$ are finite-dimensional.

This design method presupposes knowledge of the channel's second-order statistics (scattering function $C_{\mathbb{H}}(\tau,\nu)$). If $C_{\mathbb{H}}(\tau,\nu)$ is unknown or if a common pulse for a broad range of channel statistics is desired, one may use by default the brick-shaped scattering function

$$C_{\mathbb{H}}(\tau,\nu) = \begin{cases} \frac{\sigma_{\mathbb{H}}^2}{2\tau_{\max}\nu_{\max}}\,, & (\tau,\nu) \in \mathcal{R} = [0,\tau_{\max}] \times [-\nu_{\max},\nu_{\max}] \\ 0\,, & (\tau,\nu) \notin \mathcal{R}\,, \end{cases} \tag{14}$$

with a suitable (worst-case) choice of $\tau_{\max}$ and $\nu_{\max}$.

We note that a similar method can be used to optimize the transmit pulse $g(t)$ for a prescribed receive pulse $\gamma^{(0)}(t)$.

5.3. **Joint Optimization of Transmit and Receive Pulses.** With the linear optimization method discussed above, one of the two pulses must be chosen beforehand and is not optimized. A further reduction of ISI/ICI may be achieved by a *joint* optimization of the transmit pulse $g(t)$ and receive pulse $\gamma(t)$. As a cost function, we use the reciprocal of the signal-to-interference ratio (SIR) $\sigma_{\mathrm{D}}^2/\sigma_{\mathrm{I}}^2$,

$$J(g,\gamma) \,\triangleq\, \frac{\sigma_{\mathrm{I}}^2}{\sigma_{\mathrm{D}}^2} \,=\, \frac{\int_\tau \int_\nu Q_{\mathbb{H}}(\tau,\nu)\, |A_{\gamma,g}(\tau,\nu)|^2\, d\tau\, d\nu}{\int_\tau \int_\nu C_{\mathbb{H}}(\tau,\nu)\, |A_{\gamma,g}(\tau,\nu)|^2\, d\tau\, d\nu} \tag{15}$$

(cf. (10), (11)). For the sake of increased design freedom and, thus, lower ISI/ICI power, we perform the minimization of $J(g,\gamma)$ with respect to $g(t)$ and $\gamma(t)$ *without* the biorthogonality constraint (4). This means that some residual ISI/ICI will be obtained for an ideal (nondispersive) channel. Such a channel rarely occurs in practice, however. Besides, in our simulations we observed that the jointly optimized pulses tend to be almost biorthogonal (in fact, almost orthogonal).

To minimize the nonquadratic cost function $J(g,\gamma)$, numerical techniques can be used. The resulting pulses typically correspond to a local minimum of $J(g,\gamma)$ that depends on the initialization of the minimization procedure.

6. Numerical Results

We show some numerical results to demonstrate the performance of our pulse optimization techniques. We first designed an MC system, denoted $\mathcal{S}_{\mathrm{I}}$, using the linear optimization method of Section 5.2. The transmit pulse $g^{(0)}(t)$ was chosen as a truncated orthogonalized version of a Gaussian pulse for which the ratio of its root-mean-square (RMS) duration and RMS bandwidth was equal to T/F (cf. [7, 17]). The optimal biorthogonal receive pulse $\gamma^{\mathrm{opt}}(t)$ was determined according to (13). Furthermore, we designed another MC system $\mathcal{S}_{\mathrm{II}}$ using the nonlinear optimization of Section 5.3. Here, the truncated orthogonalized Gaussian pulse served to initialize both the transmit pulse and the receive pulse in the numerical optimization procedure (we used the unconstrained minimization function *fminunc* from MATLAB's optimization toolbox [29]). We compared our two optimized designs with a conventional state-of-the-art CP-OFDM system (denoted $\mathcal{S}_{\mathrm{cp}}$). As a measure of performance, we used the SIR, i.e., the reciprocal of $J(g,\gamma) = \sigma_{\mathrm{I}}^2/\sigma_{\mathrm{D}}^2$ in (15).

The system parameters were chosen similar to parameters defined by the *terrestrial digital video broadcasting (DVB-T)* standard [30]: carrier frequency 800 MHz, subcarrier separation $F = 2$ kHz, and symbol period $T = 562.5\,\mu$s. We thus have $TF = 1.125$, which corresponds to a redundancy of 12.5%. For the CP-OFDM system, this results in a CP length of $T_{\rm cp} = 62.5\,\mu$s. The random time-varying wireless channel was simulated based on the WSSUS assumption, using the brick-shaped scattering function (14) with maximum delay $\tau_{\max} \in \{20.83\,\mu\text{s}, 31.25\,\mu\text{s}, 62.5\,\mu\text{s}, 83.33\,\mu\text{s}\}$ and maximum Doppler $\nu_{\max} \in \{37\,\text{Hz}, 74\,\text{Hz}, 111\,\text{Hz}, 148\,\text{Hz}\}$. We note that maximum delay $\tau_{\max} = 83.33\,\mu$s corresponds to a maximum propagation path length of 25 km, and maximum Doppler $\nu_{\max} = 148$ Hz corresponds to a relative velocity of 200 km/h. Such path lengths and velocities may occur e.g. in single-frequency DVB-T networks providing digital video in high-speed trains.

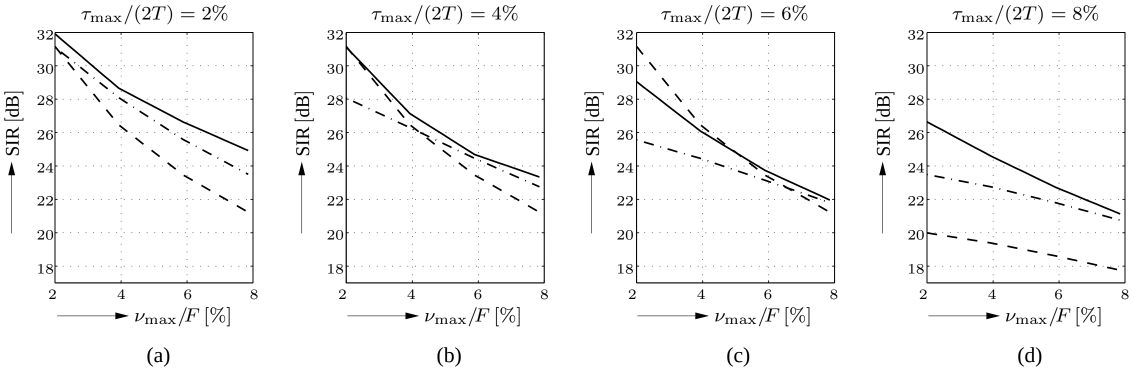

Figure 1. *SIR versus normalized maximum Doppler frequency obtained with $\mathcal{S}_{\rm cp}$ (dashed line), $\mathcal{S}_{\rm I}$ (dash-dotted line), and $\mathcal{S}_{\rm II}$ (solid line) for (a) $\tau_{\max}/(2T) = 2\%$, (b) $\tau_{\max}/(2T) = 4\%$, (c) $\tau_{\max}/(2T) = 6\%$, (d) $\tau_{\max}/(2T) = 8\%$.*

Fig. 1 shows the SIR obtained with the three MC systems $\mathcal{S}_{\rm cp}$, $\mathcal{S}_{\rm I}$, and $\mathcal{S}_{\rm II}$ as a function of the normalized maximum Doppler frequency $\nu_{\max}/F$ for four different normalized maximum delays $\tau_{\max}/T$ (we note that the optimization of the MC pulses was redone for each set of channel parameters $\tau_{\max}, \nu_{\max}$). It is seen that the SIR of all systems decreases—or, equivalently, the ISI/ICI increases—for increasing $\tau_{\max}$ and $\nu_{\max}$, except for the system $\mathcal{S}_{\rm cp}$ whose SIR is independent of $\tau_{\max}$ as long as $\tau_{\max}$ does not surpass the CP length $T_{\rm cp}$. It can furthermore be seen that our optimized pulse-shaping MC systems $\mathcal{S}_{\rm I}$ and $\mathcal{S}_{\rm II}$ outperform the CP-OFDM system $\mathcal{S}_{\rm cp}$ for a broad range of channel parameters, especially for channels with large $\nu_{\max}$ and small-to-medium $\tau_{\max}$ (see Fig. 1(a) and (b)) as well as for channels with $\tau_{\max} > T_{\rm cp}$ (see Fig. 1(d)). In particular, $\mathcal{S}_{\rm cp}$ is (slightly) superior to $\mathcal{S}_{\rm II}$ only for channels with $\tau_{\max}$ slightly below $T_{\rm cp}$ and small $\nu_{\max}$.

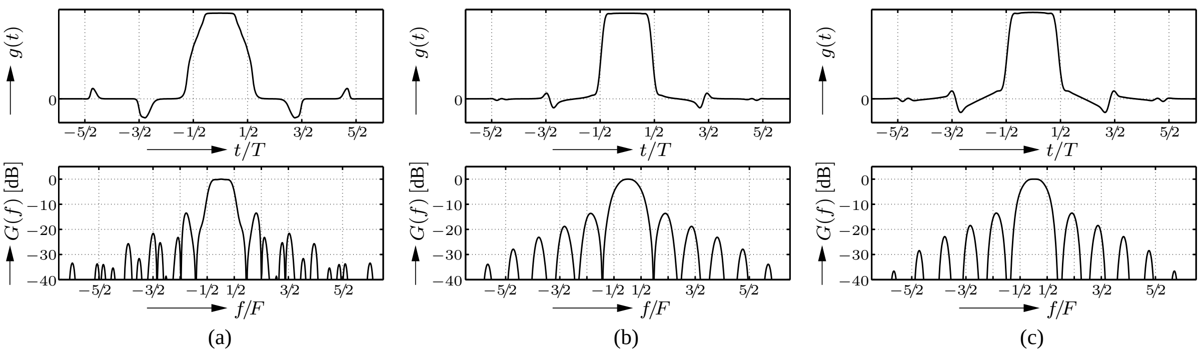

Figure 2. *Pulse shape (top) and Fourier transform magnitude (bottom, in dB) of transmit pulses obtained with joint nonlinear optimization for ($\tau_{\max}$,$\nu_{\max}$) equal to (a) (20.83 μs, 148 Hz), (b) (62.5 μs, 111 Hz), and (c) (83.33 μs, 37 Hz).*

Fig. 2 shows the transmit pulse of system $\mathcal{S}_{\rm II}$ (i.e., the result of our nonlinear numerical optimization) for three different sets of channel parameters, namely, for $(\tau_{\max}, \nu_{\max})$ equal to (20.83 μs, 148 Hz), (62.5 μs, 111 Hz), and (83.33 μs, 37 Hz). It can be seen that the pulse tends to be more concentrated in frequency when $\nu_{\max}$ is larger. This makes sense because a better frequency concentration means that the (large) Doppler shifts do not result in excessive ICI. On the other hand, for large $\tau_{\max}$ the pulse is better concentrated in time, which again makes sense since it means that the (large) delay spread does not result in excessive ISI.

7. Conclusion

For wireless communications over fast varying channels, pulse-shaping multicarrier (MC) techniques have important advantages over conventional MC transmission using CP-OFDM, because pulse shaping avoids the poor spectral concentration of the rectangular pulse employed by CP-OFDM. To leverage these advantages—most notably reduced intersymbol/intercarrier interference (ISI/ICI)—, it is necessary to design the transmit and receive pulses in accordance with the statistical properties of the wireless channel.

We applied frame theory and related mathematical models and results to the analysis and design of pulse-shaping MC systems transmitting over doubly dispersive fading channels. We demonstrated that for low ISI/ICI, the transmit and receive pulses should be jointly well localized both in time and frequency. The existence of such jointly localized pulse pairs was proven by adapting recent mathematical results on Weyl-Heisenberg frames to the MC context. This adaptation was based on the Ron-Shen duality of Weyl-Heisenberg frames and Weyl-Heisenberg Riesz sequences.

We furthermore proposed two pulse optimization techniques that minimize the ISI/ICI power for given channel statistics. One technique assumes one of the two pulses to be given and optimizes the respective other pulse, while the other technique performs a joint optimization of both pulses. Simulation results demonstrated that pulse-shaping MC systems using the optimized pulses can significantly outperform conventional CP-OFDM systems for realistic spectral efficiencies and channel parameters. Further advantages of the optimized pulses (not discussed here) include their reduced out-of-band energy, which allows a larger number of subcarriers to be used for a prescribed spectral mask, and improved robustness to frequency offsets (which act like an additional Doppler shift) and to large channel delays (which in CP-OFDM systems would require time-domain equalizers).

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