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4.3 THE COVARIANCE THEORY OF TIME-FREQUENCY ANALYSIS[0]

4.3.1 The Covariance Principle

Many important classes of linear and bilinear/quadratic time-frequency representations (TFRs) can be defined by a *covariance property*. For example, the family of *short-time Fourier transforms* [3] [9]

$$F_x^h(t,f) \,=\, \int_{-\infty}^{\infty} x(t')\, h^*(t'-t)\, e^{-j2\pi f t'}\, dt' \tag{4.3.1}$$

(where t and f denote time and frequency, respectively, $x(t)$ is the signal under analysis, and $h(t)$ is a function that does not depend on $x(t)$) consists of all linear TFRs L that are *covariant* to time-frequency (TF) shifts according to

$$L_{\mathbf{S}_{\tau,\nu}x}(t,f) \,=\, e^{-j2\pi(f-\nu)\tau}\, L_x(t-\tau, f-\nu)\,. \tag{4.3.2}$$

Here, $\mathbf{S}_{\tau,\nu}$ is the TF shift operator defined as $\big(\mathbf{S}_{\tau,\nu}x\big)(t) = x(t-\tau)\, e^{j2\pi\nu t}$. Thus, among all linear TFRs, the short-time Fourier transform is axiomatically defined by the TF shift covariance property (4.3.2). Similarly, *Cohen's class* (with signal-independent kernel $h(t_1,t_2)$), given by [3]

$$C_{x,y}(t,f) \,=\, \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, y^*(t_2)\; h^*(t_1-t, t_2-t)\; e^{-j2\pi f(t_1-t_2)}\, dt_1 dt_2\,, \tag{4.3.3}$$

comprises all bilinear/quadratic TFRs B that are covariant to TF shifts according to

$$B_{\mathbf{S}_{\tau,\nu}x,\mathbf{S}_{\tau,\nu}y}(t,f) \,=\, B_{x,y}(t-\tau, f-\nu)\,. \tag{4.3.4}$$

Similar covariance-based interpretations and definitions can be given for many other important classes of linear TFRs (e.g., wavelet transform, hyperbolic wavelet transform, and power wavelet transform [7] [9] [10]) as well as bilinear/quadratic TFRs (e.g., affine, hyperbolic, and power classes [2] [3] [6] [7] [10]; see also Articles 5.6, 7.1, and 15.3).

In this article, we present a unified *covariance theory of TF analysis* that allows the systematic construction of covariant TFRs [4] [5] [12]. (See [8] for a much more detailed treatment.) Covariance properties are important in TF analysis since specific unitary signal transformations often occur in practice—e.g., time shifts and frequency shifts as described by the TF shift operator $\mathbf{S}_{\tau,\nu}$ correspond to the delays and Doppler shifts, respectively, encountered in radar and mobile communications.

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4.3.2 Time-Frequency Displacement Operators

A key element of our covariance theory is the concept of *TF displacement operators* (DOs), which are operators that displace signals in the TF plane. The DO concept generalizes the TF shift operator $\mathbf{S}_{\tau,\nu}$, for which the TF displacements are simple shifts (translations), to other types or geometries of TF displacements.

Group fundamentals. In what follows, we will need some fundamentals of groups. A set $\mathcal{G}$ together with a binary operation $\star$ that maps $\mathcal{G}\times\mathcal{G}$ to $\mathcal{G}$ is called a *group* if it satisfies the following properties:

1. There exists an *identity element* $g_0 \in \mathcal{G}$ such that $g \star g_0 = g_0 \star g = g$ for all $g \in \mathcal{G}$.

2. To every $g \in \mathcal{G}$, there exists an *inverse element* $g^{-1} \in \mathcal{G}$ such that $g \star g^{-1} = g^{-1} \star g = g_0$.

3. Associative law: $g_1 \star (g_2 \star g_3) = (g_1 \star g_2) \star g_3$ for all $g_1, g_2, g_3 \in \mathcal{G}$.

If, in addition, $g_1 \star g_2 = g_2 \star g_1$ for all $g_1, g_2 \in \mathcal{G}$, the group is called *commutative* or *abelian*. An elementary example of a commutative group is $(\mathbb{R}, +)$ for which $g_1 \star g_2 = g_1 + g_2$, $g_0 = 0$, and $g^{-1} = -g$. Two groups $(\mathcal{G}, \star)$ and $(\mathcal{H}, \diamond)$ are said to be *isomorphic* if there exists an invertible mapping $\psi : \mathcal{G} \to \mathcal{H}$ such that $\psi(g_1 \star g_2) = \psi(g_1) \diamond \psi(g_2)$ for all $g_1, g_2 \in \mathcal{G}$.

Definition and examples of DOs. We are now ready to give a formal definition of DOs. A DO is a family of unitary operators $\mathbf{D}_{\alpha,\beta}$ indexed by a 2-D "displacement parameter" (α, β) that belongs to some group $(\mathcal{D}, \circ)$. This operator family $\{\mathbf{D}_{\alpha,\beta}\}_{(\alpha,\beta)\in\mathcal{D}}$ is supposed to satisfy the following two properties:

1. A displacement by the group identity parameter (α_0, β_0) is no displacement, i.e.,

$$\mathbf{D}_{\alpha_0,\beta_0} = \mathbf{I}\,,$$

   where $\mathbf{I}$ is the identity operator.

2. A displacement by (α_1, β_1) followed by a displacement by (α_2, β_2) is equivalent (up to a phase factor) to a single displacement by $(\alpha_1, \beta_1) \circ (\alpha_2, \beta_2)$, i.e.,

$$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = e^{j\phi(\alpha_1,\beta_1;\alpha_2,\beta_2)}\,\mathbf{D}_{(\alpha_1,\beta_1)\circ(\alpha_2,\beta_2)} \quad \forall\ (\alpha_1,\beta_1), (\alpha_2,\beta_2) \in \mathcal{D}\,, \tag{4.3.5}$$

   with $\phi(\alpha_1, \beta_1; \alpha_2, \beta_2)$ being a continuous function.

More precisely, stated in mathematical terms, a DO $\mathbf{D}_{\alpha,\beta}$ is an *irreducible and faithful projective representation* of a group $(\mathcal{D}, \circ)$; the function $e^{j\phi(\alpha_1,\beta_1;\alpha_2,\beta_2)}$ is known as the *cocycle*. For $e^{j\phi(\alpha_1,\beta_1;\alpha_2,\beta_2)} \equiv 1$, $\mathbf{D}_{\alpha,\beta}$ is a *unitary* group representation [8].

Two basic examples of a DO are the following:

- The TF shift operator $\mathbf{S}_{\tau,\nu}$. Here, $(\alpha,\beta) = (\tau,\nu)$ and $(\mathcal{D},\circ)$ is the commutative group $(\mathbb{R}^2,+)$ with operation $(\tau_1,\nu_1)\circ(\tau_2,\nu_2) = (\tau_1+\tau_2,\nu_1+\nu_2)$; furthermore $(\tau_0,\nu_0) = (0,0)$, $(\tau,\nu)^{-1} = (-\tau,-\nu)$, and $\phi(\tau_1,\nu_1;\tau_2,\nu_2) = -2\pi\nu_1\tau_2$.

- The TF scaling/time shift operator $\mathbf{R}_{\sigma,\tau}$ defined as $\big(\mathbf{R}_{\sigma,\tau}x\big)(t) = \frac{1}{\sqrt{|\sigma|}}\, x\big(\frac{t-\tau}{\sigma}\big)$. Here, $(\alpha,\beta) = (\sigma,\tau)$, $(\mathcal{D},\circ)$ is the noncommutative *affine group* with $\mathcal{D} = \mathbb{R}^+ \times \mathbb{R}$ and group operation $(\sigma_1,\tau_1)\circ(\sigma_2,\tau_2) = (\sigma_1\sigma_2,\tau_1\sigma_2+\tau_2)$; furthermore $(\sigma_0,\tau_0) = (1,0)$, $(\sigma,\tau)^{-1} = (1/\sigma,-\tau/\sigma)$, and $\phi(\sigma_1,\tau_1;\sigma_2,\tau_2) \equiv 0$.

Additional structure of DOs. The interpretation that a DO $\mathbf{D}_{\alpha,\beta}$ performs TF displacements motivates certain topological assumptions which can be shown [8] to imply that $(\mathcal{D},\circ)$ is a *simply connected 2-D Lie group*. This, in turn, can be shown to have the following two important consequences [8]:

1. The group $(\mathcal{D},\circ)$ underlying $\mathbf{D}_{\alpha,\beta}$ is either isomorphic to the group $(\mathbb{R}^2,+)$ underlying $\mathbf{S}_{\tau,\nu}$ or isomorphic to the affine group underlying $\mathbf{R}_{\sigma,\tau}$ (see the examples above).

2. The DO $\mathbf{D}_{\alpha,\beta}$ is *separable* (possibly up to a phase factor) in the following sense: there exists an isomorphism $\psi : (\alpha,\beta) \to (\alpha',\beta')$ such that the parameter-transformed DO $\mathbf{D}_{\psi^{-1}(\alpha',\beta')}$ (briefly written as $\mathbf{D}_{\alpha',\beta'}$ hereafter) can be factored as [8]

$$\mathbf{D}_{\alpha',\beta'} = e^{j\mu(\alpha',\beta')}\, \mathbf{B}_{\beta'}\, \mathbf{A}_{\alpha'} \,. \tag{4.3.6}$$

   Here, $\mathbf{A}_{\alpha'}$ and $\mathbf{B}_{\beta'}$ (termed *partial DOs*) are indexed by 1-D displacement parameters $\alpha' \in (\mathcal{A},\bullet)$ and $\beta' \in (\mathcal{B},*)$, respectively, where $(\mathcal{A},\bullet)$ and $(\mathcal{B},*)$ are *commutative* groups that are isomorphic to $(\mathbb{R},+)$. For example,

$$\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau \qquad \text{and} \qquad \mathbf{R}_{\sigma,\tau} = \mathbf{T}_\tau \mathbf{C}_\sigma \,,$$

   with the time-shift operator $\mathbf{T}_\tau$, frequency-shift operator $\mathbf{F}_\nu$, and TF scaling operator $\mathbf{C}_\sigma$ defined as $(\mathbf{T}_\tau x)(t) = x(t-\tau)$, $(\mathbf{F}_\nu x)(t) = x(t)\, e^{j2\pi\nu t}$, and $(\mathbf{C}_\sigma x)(t) = \frac{1}{\sqrt{|\sigma|}}\, x\big(\frac{t}{\sigma}\big)$, respectively.

4.3.3 Covariant Signal Representations: Group Domain

We shall now discuss the construction of TF representations that are covariant to a given DO $\mathbf{D}_{\alpha,\beta}$. This construction is a two-stage process: first, we construct covariant signal representations that are functions of the displacement parameter (i.e., the group variables) (α,β). Subsequently (in Sections 4.3.4 and 4.3.5), we will convert these covariant (α,β) representations into covariant TF representations.

Covariance in the group domain. A linear (α, β) representation $L_x(\alpha, \beta)$ is called *covariant to a DO* $\mathbf{D}_{\alpha,\beta}$ if

$$L_{\mathbf{D}_{\alpha',\beta'}x}(\alpha, \beta) = e^{j\phi\left((\alpha,\beta)\circ(\alpha',\beta')^{-1};\, \alpha',\beta'\right)} L_x\big((\alpha, \beta) \circ (\alpha', \beta')^{-1}\big)\,, \tag{4.3.7}$$

for all signals $x(t)$ and for all $(\alpha, \beta), (\alpha', \beta') \in \mathcal{D}$ [8]. Similarly, a bilinear/quadratic (α, β) representation $B_{x,y}(\alpha, \beta)$ is called covariant to a DO $\mathbf{D}_{\alpha,\beta}$ if

$$B_{\mathbf{D}_{\alpha',\beta'}x,\, \mathbf{D}_{\alpha',\beta'}y}(\alpha, \beta) = B_{x,y}\big((\alpha, \beta) \circ (\alpha', \beta')^{-1}\big)\,, \tag{4.3.8}$$

for all signal pairs $x(t), y(t)$ and for all $(\alpha, \beta), (\alpha', \beta') \in \mathcal{D}$ [8] [12]. Note that the "linear" covariance property (4.3.7) differs from the "bilinear" covariance property (4.3.8) in that it contains a phase factor.

For example, for $\mathbf{S}_{\tau,\nu}$ the covariance properties (4.3.7) and (4.3.8) become

$$L_{\mathbf{S}_{\tau',\nu'}x}(\tau, \nu) = e^{-j2\pi(\nu-\nu')\tau'} L_x(\tau-\tau', \nu-\nu')$$
$$B_{\mathbf{S}_{\tau',\nu'}x,\, \mathbf{S}_{\tau',\nu'}y}(\tau, \nu) = B_{x,y}(\tau-\tau', \nu-\nu')\,,$$

which are seen to be identical to (4.3.2) and (4.3.4), respectively. For $\mathbf{R}_{\sigma,\tau}$, we obtain

$$L_{\mathbf{R}_{\sigma',\tau'}x}(\sigma, \tau) = L_x\Big(\frac{\sigma}{\sigma'}, \frac{\tau-\tau'}{\sigma'}\Big)$$
$$B_{\mathbf{R}_{\sigma',\tau'}x,\, \mathbf{R}_{\sigma',\tau'}y}(\sigma, \tau) = B_{x,y}\Big(\frac{\sigma}{\sigma'}, \frac{\tau-\tau'}{\sigma'}\Big).$$

Construction of covariant (α, β) representations. It can be shown [8] that all linear (α, β) representations covariant to a DO $\mathbf{D}_{\alpha,\beta}$ as defined in (4.3.7) are given by

$$L_x(\alpha, \beta) = \langle x, \mathbf{D}_{\alpha,\beta} h\rangle = \int_{\mathcal{I}} x(t)\, (\mathbf{D}_{\alpha,\beta} h)^*(t)\, dt\,, \tag{4.3.9}$$

where $h(t)$ is an arbitrary function and $\mathcal{I}$ is the time interval on which $\mathbf{D}_{\alpha,\beta}$ is defined. Similarly, all bilinear/quadratic (α, β) representations covariant to a DO $\mathbf{D}_{\alpha,\beta}$ as defined in (4.3.8) are given by [8] [12]

$$B_{x,y}(\alpha, \beta) = \langle x, \mathbf{D}_{\alpha,\beta}\mathbf{H}\mathbf{D}_{\alpha,\beta}^{-1} y\rangle = \int_{\mathcal{I}}\int_{\mathcal{I}} x(t_1)\, y^*(t_2)\, \big[\mathbf{D}_{\alpha,\beta}\mathbf{H}\mathbf{D}_{\alpha,\beta}^{-1}\big]^*(t_1, t_2)\, dt_1 dt_2\,, \tag{4.3.10}$$

where $\mathbf{H}$ is an arbitrary linear operator and $\big[\mathbf{D}_{\alpha,\beta}\mathbf{H}\mathbf{D}_{\alpha,\beta}^{-1}\big](t_1, t_2)$ denotes the kernel of the composed operator $\mathbf{D}_{\alpha,\beta}\mathbf{H}\mathbf{D}_{\alpha,\beta}^{-1}$. The equations (4.3.9) and (4.3.10) provide canonical expressions for all covariant linear and bilinear/quadratic (α, β) representations.

For example, for $\mathbf{S}_{\tau,\nu}$ these expressions yield

$$L_x(\tau, \nu) = \int_{-\infty}^{\infty} x(t)\, h^*(t-\tau)\, e^{-j2\pi\nu t}\, dt$$
$$B_{x,y}(\tau, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, y^*(t_2)\, h^*(t_1-\tau, t_2-\tau)\, e^{-j2\pi\nu(t_1-t_2)}\, dt_1 dt_2\,,$$

which are seen to be the short-time Fourier transform in (4.3.1) and Cohen's class in (4.3.3), respectively. Similarly, for $\mathbf{R}_{\sigma,\tau}$ we obtain time-scale versions of the wavelet transform and the affine class [3]:

$$L_x(\sigma,\tau) = \frac{1}{\sqrt{|\sigma|}} \int_{-\infty}^{\infty} x(t)\, h^*\Big(\frac{t-\tau}{\sigma}\Big)\, dt$$

$$B_{x,y}(\sigma,\tau) = \frac{1}{|\sigma|} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, y^*(t_2)\; h^*\Big(\frac{t_1-\tau}{\sigma}, \frac{t_2-\tau}{\sigma}\Big)\, dt_1 dt_2\,.$$

4.3.4 The Displacement Function

The covariant (α,β) representations constructed above can be converted into covariant TF representations (ultimately, we are interested in TF representations and not in (α,β) representations). This conversion uses a mapping $(\alpha,\beta)\to(t,f)$ that is termed the *displacement function* (DF) since it describes the TF displacements performed by a DO $\mathbf{D}_{\alpha,\beta}$ in terms of TF coordinates [8].

The DF concept is based on the following reasoning. If a signal $x(t)$ is TF localized about some TF point (t_1,f_1), then the transformed ("displaced") signal $(\mathbf{D}_{\alpha,\beta}\, x)(t)$ will be localized about some other TF point (t_2,f_2) that depends on (t_1,f_1) and α,β. We can thus write

$$(t_2,f_2) = e_{\mathbf{D}}(t_1,f_1;\alpha,\beta)\,,$$

with some function $e_{\mathbf{D}}(t,f;\alpha,\beta)$ that will be called the *extended DF* of the DO $\mathbf{D}_{\alpha,\beta}$. For $\mathbf{S}_{\tau,\nu}$ and $\mathbf{R}_{\sigma,\tau}$, for example, it can easily be argued that the extended DF is given by

$$e_{\mathbf{S}}(t,f;\tau,\nu) = (t+\tau, f+\nu)\,, \qquad e_{\mathbf{R}}(t,f;\sigma,\tau) = \Big(\sigma t+\tau, \frac{f}{\sigma}\Big)\,. \tag{4.3.11}$$

Construction of the extended DF. In general, the extended DF cannot be found "by inspection," and therefore we need a systematic method for constructing the extended DF of a given DO $\mathbf{D}_{\alpha,\beta}$ [8]. The expression[1] $\mathbf{D}_{\alpha,\beta} = e^{j\mu(\alpha,\beta)}\,\mathbf{B}_\beta\mathbf{A}_\alpha$ (see (4.3.6)) states that $\mathbf{D}_{\alpha,\beta}$ is, up to a phase factor, the composition or series connection of $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$. Hence, $e_{\mathbf{D}}(t,f;\alpha,\beta)$ can be obtained by composing the extended DF of the partial DO $\mathbf{A}_\alpha$, $e_{\mathbf{A}}(t,f;\alpha)$, and the extended DF of the partial DO $\mathbf{B}_\beta$, $e_{\mathbf{B}}(t,f;\beta)$, according to

$$e_{\mathbf{D}}(t,f;\alpha,\beta) \;=\; e_{\mathbf{B}}\big(e_{\mathbf{A}}(t,f;\alpha);\,\beta\big)\,. \tag{4.3.12}$$

Using this expression, the task of constructing $e_{\mathbf{D}}(t,f;\alpha,\beta)$ reduces to the task of constructing $e_{\mathbf{A}}(t,f;\alpha)$ and $e_{\mathbf{B}}(t,f;\beta)$. We will explain the construction of $e_{\mathbf{A}}(t,f;\alpha)$ [8]; the construction of $e_{\mathbf{B}}(t,f;\beta)$ is of course analogous.

[1]For simplicity of notation, we assume that the parameter transformation $\psi: (\alpha,\beta)\to(\alpha',\beta')$ described in Section 4.3.2 has already been performed, and we write α,β instead of α',β'.

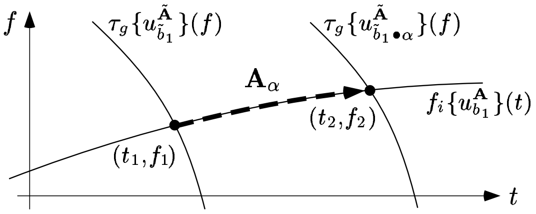

Figure 4.1: Construction of the extended DF of $\mathbf{A}_\alpha$.

We first recall the "definition" of $e_{\mathbf{A}}(t, f; \alpha)$: If $x(t)$ is localized about (t_1, f_1), then $(\mathbf{A}_\alpha x)(t)$ will be localized about $(t_2, f_2) = e_{\mathbf{A}}(t_1, f_1; \alpha)$. In order to find (t_2, f_2), we consider the (generalized) eigenfunctions $u_b^{\mathbf{A}}(t)$ of $\mathbf{A}_\alpha$. These are defined by $\big(\mathbf{A}_\alpha u_b^{\mathbf{A}}\big)(t) \propto u_b^{\mathbf{A}}(t)$ and indexed by a parameter $b \in (\tilde{\mathcal{A}}, \tilde{\bullet})$, where $(\tilde{\mathcal{A}}, \tilde{\bullet})$ is again a commutative group isomorphic to $(\mathbb{R}, +)$. The TF locus of $u_b^{\mathbf{A}}(t)$ is characterized by the instantaneous frequency $f_i\{u_b^{\mathbf{A}}\}(t)$ or the group delay $\tau_g\{u_b^{\mathbf{A}}\}(f)$, whichever exists.[2] Here, e.g., we assume existence of $f_i\{u_b^{\mathbf{A}}\}(t)$. Let us choose b_1 such that the TF curve defined by $f_i\{u_{b_1}^{\mathbf{A}}\}(t)$ passes through (t_1, f_1), i.e.,

$$f_i\{u_{b_1}^{\mathbf{A}}\}(t_1) = f_1 \,. \tag{4.3.13}$$

This is shown in Fig. 4.1. Now since $\big(\mathbf{A}_\alpha u_b^{\mathbf{A}}\big)(t) \propto u_b^{\mathbf{A}}(t)$, $\mathbf{A}_\alpha$ preserves the TF locus of $u_{b_1}^{\mathbf{A}}(t)$. Therefore, *under the action of* $\mathbf{A}_\alpha$*, all TF points on the curve* $f_i\{u_{b_1}^{\mathbf{A}}\}(t)$*—including* (t_1, f_1)*—are mapped again onto TF points on* $f_i\{u_{b_1}^{\mathbf{A}}\}(t)$. Hence, $(t_2, f_2) = e_{\mathbf{A}}(t_1, f_1; \alpha)$ must lie on $f_i\{u_{b_1}^{\mathbf{A}}\}(t)$ (see Fig. 4.1), i.e., there must be

$$f_i\{u_{b_1}^{\mathbf{A}}\}(t_2) = f_2 \,. \tag{4.3.14}$$

In order to find the exact position of (t_2, f_2) on the TF curve defined by $f_i\{u_{b_1}^{\mathbf{A}}\}(t)$, we use the fact that to any partial displacement operator $\mathbf{A}_\alpha$ there exists a *dual operator* $\tilde{\mathbf{A}}_{\tilde{\alpha}}$ with $\tilde{\alpha} \in (\tilde{\mathcal{A}}, \tilde{\bullet})$ that is defined by the "almost commutation relation" $\tilde{\mathbf{A}}_{\tilde{\alpha}} \mathbf{A}_\alpha = e^{j2\pi\psi_{\mathcal{A}}(\alpha)\psi_{\tilde{\mathcal{A}}}(\tilde{\alpha})}\, \mathbf{A}_\alpha \tilde{\mathbf{A}}_{\tilde{\alpha}}$ [8] [11]. For example, the dual operator of $\mathbf{T}_\tau$ is $\mathbf{F}_\nu$ and vice versa. Let $u_{\tilde{b}}^{\tilde{\mathbf{A}}}(t)$ with $\tilde{b} \in (\mathcal{A}, \bullet)$ denote the (generalized) eigenfunctions of $\tilde{\mathbf{A}}_{\tilde{\alpha}}$ and assume, e.g., that the group delay $\tau_g\{u_{\tilde{b}}^{\tilde{\mathbf{A}}}\}(f)$ exists. Let us choose $\tilde{b}_1$ such that the TF curve defined by $\tau_g\{u_{\tilde{b}_1}^{\tilde{\mathbf{A}}}\}(f)$ passes through (t_1, f_1), i.e. (see Fig. 4.1)

$$\tau_g\{u_{\tilde{b}_1}^{\tilde{\mathbf{A}}}\}(f_1) = t_1 \,. \tag{4.3.15}$$

Now assuming suitable parameterization of $u_{\tilde{b}}^{\tilde{\mathbf{A}}}(t)$, it can be shown [8] that

$$\big(\mathbf{A}_\alpha u_{\tilde{b}}^{\tilde{\mathbf{A}}}\big)(t) \propto u_{\tilde{b}\bullet\alpha}^{\tilde{\mathbf{A}}}(t) \qquad \text{for all } \alpha, \tilde{b} \in (\mathcal{A}, \bullet) \,. \tag{4.3.16}$$

[2]The instantaneous frequency of a signal $x(t)$ is defined as $f_i\{x\}(t) = \frac{1}{2\pi}\frac{d}{dt}\arg\{x(t)\}$; it exists if $\arg\{x(t)\}$ is differentiable and $x(t) \neq 0$ almost everywhere. The group delay of $x(t)$ is defined as $\tau_g\{x\}(f) = -\frac{1}{2\pi}\frac{d}{df}\arg\{X(f)\}$ with $X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt$; it exists if $\arg\{X(f)\}$ is differentiable and $X(f) \neq 0$ almost everywhere.

Thus, $\mathbf{A}_\alpha$ *maps all TF points on* $\tau_g\{u_{\tilde{b}_1}^{\tilde{\mathbf{A}}}\}(f)$*—including* (t_1, f_1)*—onto TF points on* $\tau_g\{u_{\tilde{b}_1 \tilde{\bullet} \alpha}^{\tilde{\mathbf{A}}}\}(f)$. So $(t_2, f_2) = e_{\mathbf{A}}(t_1, f_1; \alpha)$ must lie on $\tau_g\{u_{\tilde{b}_1 \tilde{\bullet} \alpha}^{\tilde{\mathbf{A}}}\}(f)$ (see Fig. 4.1), i.e.,

$$\tau_g\{u_{\tilde{b}_1 \tilde{\bullet} \alpha}^{\tilde{\mathbf{A}}}\}(f_2) = t_2 \,. \tag{4.3.17}$$

The construction of $e_{\mathbf{A}}$ can now be summarized as follows (see Fig. 4.1):

1. For any given (t_1, f_1), we calculate associated eigenfunction parameters $b_1 \in (\tilde{\mathcal{A}}, \tilde{\bullet})$ and $\tilde{b}_1 \in (\mathcal{A}, \bullet)$ as the solutions to (4.3.13) and (4.3.15), respectively:

$$f_i\{u_{b_1}^{\mathbf{A}}\}(t_1) = f_1 \,, \qquad \tau_g\{u_{\tilde{b}_1}^{\tilde{\mathbf{A}}}\}(f_1) = t_1 \,. \tag{4.3.18}$$

2. The extended DF $e_{\mathbf{A}}$ is defined by the identity $(t_2, f_2) \equiv e_{\mathbf{A}}(t_1, f_1; \alpha)$, where (t_2, f_2) is obtained as the solution to the system of equations (4.3.14), (4.3.17):

$$f_i\{u_{b_1}^{\mathbf{A}}\}(t_2) = f_2 \,, \qquad \tau_g\{u_{\tilde{b}_1 \tilde{\bullet} \alpha}^{\tilde{\mathbf{A}}}\}(f_2) = t_2 \,. \tag{4.3.19}$$

A similar construction of $e_{\mathbf{A}}$ can be used if, e.g., $\tau_g\{u_b^{\mathbf{A}}\}(f)$ and $f_i\{u_{\tilde{b}}^{\tilde{\mathbf{A}}}\}(t)$ exist instead of $f_i\{u_b^{\mathbf{A}}\}(t)$ and $\tau_g\{u_{\tilde{b}}^{\tilde{\mathbf{A}}}\}(f)$. An example for this case will be provided in Section 4.3.6.

After construction of the extended DF of $\mathbf{A}_\alpha$ as detailed above, the extended DF of $\mathbf{B}_\beta$ is constructed by means of an analogous procedure, and finally the extended DF of $\mathbf{D}_{\alpha,\beta}$ is obtained by composing $e_{\mathbf{A}}$ and $e_{\mathbf{B}}$ according to (4.3.12).

The DF. The above discussion has shown how to construct the *extended* DF $e_{\mathbf{D}}(t, f; \alpha, \beta)$. We go on to define the DF $d_{\mathbf{D}}(\alpha, \beta)$ by fixing t, f in $e_{\mathbf{D}}(t, f; \alpha, \beta)$:

$$d_{\mathbf{D}}(\alpha, \beta) \triangleq e_{\mathbf{D}}(t_0, f_0; \alpha, \beta) \qquad \text{with } t_0, f_0 \text{ arbitrary but fixed.}$$

The DF is a mapping $(\alpha, \beta) \to (t, f)$, i.e., from the displacement parameter (or group) domain to the TF domain. If the inverse DF $d_{\mathbf{D}}^{-1}(t, f)$ exists, then it can be shown [8] that the extended DF can be written as

$$e_{\mathbf{D}}(t, f; \alpha, \beta) \;=\; d_{\mathbf{D}}\big(d_{\mathbf{D}}^{-1}(t, f) \circ (\alpha, \beta)\big) \,. \tag{4.3.20}$$

Examples. Application of the construction explained above to the DOs $\mathbf{S}_{\tau,\nu}$ and $\mathbf{R}_{\sigma,\tau}$ yields the extended DFs $e_{\mathbf{S}}(t, f; \tau, \nu) = (t + \tau, f + \nu)$ and $e_{\mathbf{R}}(t, f; \sigma, \tau) = \big(\sigma t + \tau, \frac{f}{\sigma}\big)$. Note that this agrees with (4.3.11). Corresponding DFs are obtained by setting, e.g., $t = f = 0$ in $e_{\mathbf{S}}(t, f; \tau, \nu)$ and $t = 0$, $f = f_0 \neq 0$ in $e_{\mathbf{R}}(t, f; \sigma, \tau)$:

$$d_{\mathbf{S}}(\tau, \nu) = (\tau, \nu) \,, \qquad d_{\mathbf{R}}(\sigma, \tau) = \left(\tau, \frac{f_0}{\sigma}\right) .$$

A further example will be discussed in detail in Section 4.3.6.

4.3.5 Covariant Signal Representations: Time-Frequency Domain

In Section 4.3.3, we derived covariant linear and bilinear/quadratic signal representations that were functions of the displacement parameter (α, β). Using the inverse DF mapping $(\alpha, \beta) = d_{\mathbf{D}}^{-1}(t, f)$, these covariant (α, β) representations can now be converted into covariant TF representations (TFRs).

Covariance in the TF domain. A linear TFR $\tilde{L}_x(t, f)$ is called *covariant to a DO* $\mathbf{D}_{\alpha,\beta}$ if

$$\tilde{L}_{\mathbf{D}_{\alpha',\beta'}x}(t, f) = e^{j\phi\left(d_{\mathbf{D}}^{-1}(t,f)\circ(\alpha',\beta')^{-1};\,\alpha',\beta'\right)}\, \tilde{L}_x\left(e_{\mathbf{D}}(t, f; (\alpha', \beta')^{-1})\right), \qquad (4.3.21)$$

for all $x(t)$ and for all $(\alpha', \beta') \in \mathcal{D}$ [8]. Similarly, a bilinear/quadratic TFR $\tilde{B}_{x,y}(t, f)$ is called covariant to a DO $\mathbf{D}_{\alpha,\beta}$ if

$$\tilde{B}_{\mathbf{D}_{\alpha',\beta'}x,\,\mathbf{D}_{\alpha',\beta'}y}(t, f) = \tilde{B}_{x,y}\left(e_{\mathbf{D}}(t, f; (\alpha', \beta')^{-1})\right), \qquad (4.3.22)$$

for all $x(t), y(t)$ and for all $(\alpha', \beta') \in \mathcal{D}$ [8]. With (4.3.20), it is seen that these covariance properties are simply the (α, β)-domain (group-domain) covariance properties (4.3.7) and (4.3.8) with the transformation $(\alpha, \beta) = d_{\mathbf{D}}^{-1}(t, f)$.

For example, for $\mathbf{S}_{\tau,\nu}$ the covariance properties (4.3.21) and (4.3.22) become

$$\tilde{L}_{\mathbf{S}_{\tau',\nu'}x}(t, f) = e^{-j2\pi(f-\nu')\tau'}\, \tilde{L}_x(t-\tau', f-\nu')$$

$$\tilde{B}_{\mathbf{S}_{\tau',\nu'}x,\,\mathbf{S}_{\tau',\nu'}y}(t, f) = \tilde{B}_{x,y}(t-\tau', f-\nu').$$

These relations are equivalent to (4.3.2) and (4.3.4), respectively. For $\mathbf{R}_{\sigma,\tau}$, we obtain

$$\tilde{L}_{\mathbf{R}_{\sigma',\tau'}x}(t, f) = \tilde{L}_x\left(\frac{t-\tau'}{\sigma'}, \sigma' f\right)$$

$$\tilde{B}_{\mathbf{R}_{\sigma',\tau'}x,\,\mathbf{R}_{\sigma',\tau'}y}(t, f) = \tilde{B}_{x,y}\left(\frac{t-\tau'}{\sigma'}, \sigma' f\right).$$

Construction of covariant TF representations. It has been observed above that the TF covariance properties (4.3.21) and (4.3.22) are equivalent to the (α, β)-domain covariance properties (4.3.7) and (4.3.8), respectively, apart from the transformation $(\alpha, \beta) = d_{\mathbf{D}}^{-1}(t, f)$. From this equivalence, it follows that all covariant linear TFRs $\tilde{L}_x(t, f)$ are obtained from corresponding covariant linear (α, β) representations $L_x(\alpha, \beta)$ as given by (4.3.9) simply by setting $(\alpha, \beta) = d_{\mathbf{D}}^{-1}(t, f)$. Consequently, all linear TFRs covariant to a DO $\mathbf{D}_{\alpha,\beta}$ are given by

$$\tilde{L}_x(t, f) = L_x(\alpha, \beta)\Big|_{(\alpha,\beta)=d_{\mathbf{D}}^{-1}(t,f)} = \langle x, \mathbf{D}_{d_{\mathbf{D}}^{-1}(t,f)}h\rangle = \int_{\mathcal{I}} x(t')\, (\mathbf{D}_{d_{\mathbf{D}}^{-1}(t,f)}h)^*(t')\, dt', \qquad (4.3.23)$$

where $h(t)$ is an arbitrary function and $\mathcal{I}$ is the time interval on which $\mathbf{D}_{\alpha,\beta}$ is defined. Similarly, all covariant bilinear/quadratic TFRs $\tilde{B}_{x,y}(t, f)$ are obtained from corresponding covariant bilinear/quadratic (α,β) representations $B_{x,y}(\alpha,\beta)$ as given by (4.3.10) by setting $(\alpha,\beta) = d_{\mathbf{D}}^{-1}(t,f)$. Thus, all covariant bilinear/quadratic TFRs are given by

$$\tilde{B}_{x,y}(t,f) \;=\; B_{x,y}(\alpha,\beta)\Big|_{(\alpha,\beta)=d_{\mathbf{D}}^{-1}(t,f)} \;=\; \langle x, \mathbf{D}_{d_{\mathbf{D}}^{-1}(t,f)} \mathbf{H} \mathbf{D}^{-1}_{d_{\mathbf{D}}^{-1}(t,f)} y\rangle$$
$$= \int_{\mathcal{I}}\int_{\mathcal{I}} x(t_1)\, y^*(t_2)\, \big[\mathbf{D}_{d_{\mathbf{D}}^{-1}(t,f)} \mathbf{H} \mathbf{D}^{-1}_{d_{\mathbf{D}}^{-1}(t,f)}\big]^*(t_1,t_2)\, dt_1 dt_2\,, \qquad (4.3.24)$$

where $\mathbf{H}$ is an arbitrary linear operator. The equations (4.3.23) and (4.3.24) provide canonical expressions for all covariant linear and bilinear/quadratic TFRs [8].

For example, the classes of all linear and bilinear/quadratic TFRs covariant to $\mathbf{S}_{\tau,\nu}$ follow from (4.3.23) and (4.3.24) as

$$\tilde{L}_x(t,f) \;=\; \int_{-\infty}^{\infty} x(t')\, h^*(t'-t)\, e^{-j2\pi f t'}\, dt'$$
$$\tilde{B}_{x,y}(t,f) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, y^*(t_2)\; h^*(t_1-t, t_2-t)\; e^{-j2\pi f(t_1-t_2)}\, dt_1 dt_2\,;$$

they are seen to be the short-time Fourier transform in (4.3.1) and Cohen's class in (4.3.3), respectively. Similarly, the classes of all linear and bilinear/quadratic TFRs covariant to $\mathbf{R}_{\sigma,\tau}$ are obtained as

$$\tilde{L}_x(t,f) \;=\; \sqrt{\left|\frac{f}{f_0}\right|}\int_{-\infty}^{\infty} x(t')\, h^*\!\left(\frac{f}{f_0}\,(t'-t)\right) dt'$$
$$\tilde{B}_{x,y}(t,f) \;=\; \left|\frac{f}{f_0}\right| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, y^*(t_2)\; h^*\!\left(\frac{f}{f_0}\,(t_1-t), \frac{f}{f_0}\,(t_2-t)\right) dt_1 dt_2\,;$$

they are TF versions of the wavelet transform [9] and the affine class [2] [3]. Thus, the short-time Fourier transform, the wavelet transform, Cohen's class, and the affine class have all been obtained by means of the systematic construction provided by covariance theory.

4.3.6 Example: Hyperbolic Wavelet Transform and Hyperbolic Class

So far, we have considered the elementary DOs $\mathbf{S}_{\tau,\nu}$ and $\mathbf{R}_{\sigma,\tau}$ as illustrative examples. Let us now apply our covariance theory to a situation that is a little less elementary. We consider the DO $\mathbf{V}_{\sigma,\gamma}$ defined by

$$(\mathbf{V}_{\sigma,\gamma}x)(t) \;=\; (\mathbf{G}_\gamma \mathbf{C}_\sigma x)(t) \;=\; \frac{1}{\sqrt{\sigma}}\, x\!\left(\frac{t}{\sigma}\right) e^{j2\pi\gamma \ln(t/t_0)}\,, \qquad t>0,\ \sigma>0,\ \gamma\in\mathbb{R}.$$

Here, $\mathbf{C}_\sigma$ is the TF scaling operator defined by $(\mathbf{C}_\sigma x)(t) = \frac{1}{\sqrt{\sigma}}\, x\big(\frac{t}{\sigma}\big)$ with $\sigma>0$; furthermore, $\mathbf{G}_\gamma$ is the *hyperbolic frequency-shift operator* defined by $(\mathbf{G}_\gamma x)(t) =$

$x(t)\, e^{j2\pi\gamma \ln(t/t_0)}$ with $t_0 > 0$ arbitrary but fixed. Note that $\mathbf{C}_\sigma$ and $\mathbf{G}_\gamma$ are dual operators since $\mathbf{G}_\gamma \mathbf{C}_\sigma = e^{j2\pi\gamma \ln \sigma}\, \mathbf{C}_\sigma \mathbf{G}_\gamma$. Comparison of the relation $\mathbf{V}_{\sigma_2,\gamma_2}\mathbf{V}_{\sigma_1,\gamma_1} = e^{-j2\pi\gamma_1 \ln \sigma_2}\, \mathbf{V}_{\sigma_1\sigma_2,\gamma_1+\gamma_2}$ with (4.3.5) shows that (σ,γ) belongs to the commutative group $(\mathbb{R}^+ \times \mathbb{R}, \circ)$ with group law $(\sigma_1,\gamma_1) \circ (\sigma_2,\gamma_2) = (\sigma_1\sigma_2, \gamma_1+\gamma_2)$, identity element $(1,0)$, and inverse elements $(\sigma,\gamma)^{-1} = (1/\sigma, -\gamma)$. This group is isomorphic to the group $(\mathbb{R}^2, +)$. Furthermore, we see that the cocycle phase function is given by $\phi(\sigma_1,\gamma_1;\sigma_2,\gamma_2) = -2\pi\gamma_1 \ln \sigma_2$.

We now begin our construction of TFRs covariant to the DO $\mathbf{V}_{\sigma,\gamma}$. In the (σ,γ) domain, the covariance properties (4.3.7) and (4.3.8) read as

$$L_{\mathbf{V}_{\sigma',\gamma'}x}(\sigma,\gamma) \;=\; e^{-j2\pi(\gamma-\gamma')\ln\sigma'}\, L_x\Big(\frac{\sigma}{\sigma'}, \gamma-\gamma'\Big)$$
$$B_{\mathbf{V}_{\sigma',\gamma'}x,\mathbf{V}_{\sigma',\gamma'}y}(\sigma,\gamma) \;=\; B_{x,y}\Big(\frac{\sigma}{\sigma'}, \gamma-\gamma'\Big),$$

and the covariant linear and bilinear/quadratic (σ,γ) representations are obtained from (4.3.9) and (4.3.10) as

$$L_x(\sigma,\gamma) \;=\; \frac{1}{\sqrt{\sigma}} \int_0^\infty x(t)\, h^*\Big(\frac{t}{\sigma}\Big)\, e^{-j2\pi\gamma \ln(t/t_0)}\, dt\,, \qquad \sigma > 0$$
$$B_{x,y}(\sigma,\gamma) \;=\; \frac{1}{\sigma} \int_0^\infty \int_0^\infty x(t_1)\, y^*(t_2)\, h^*\Big(\frac{t_1}{\sigma}, \frac{t_2}{\sigma}\Big)\, e^{-j2\pi\gamma \ln(t_1/t_2)}\, dt_1 dt_2\,, \qquad \sigma > 0\,.$$

Next, we construct the DF of $\mathbf{V}_{\sigma,\gamma} = \mathbf{G}_\gamma \mathbf{C}_\sigma$. We first consider the extended DF of $\mathbf{C}_\sigma$. Although clearly $e_{\mathbf{C}}(t,f;\sigma) = \big(\sigma t, f/\sigma\big)$, we shall derive $e_{\mathbf{C}}(t,f;\sigma)$ using the systematic construction procedure from Section 4.3.4. The eigenfunctions of $\mathbf{C}_\sigma$ are $u_\gamma^{\mathbf{C}}(t) = \frac{1}{\sqrt{t}}\, e^{j2\pi\gamma \ln(t/t_0)}$, with instantaneous frequency $f_i\{u_\gamma^{\mathbf{C}}\}(t) = \gamma/t$. The eigenfunctions of the dual operator $\mathbf{G}_\gamma$ are given by $u_s^{\mathbf{G}}(t) = \delta(t-s)$, with group delay $\tau_g\{u_s^{\mathbf{G}}\}(f) \equiv s$. (It can be verified that (4.3.16) is satisfied: $\big(\mathbf{C}_\sigma u_s^{\mathbf{G}}\big)(t) = \sqrt{\sigma}\, \delta(t-\sigma s) \propto u_{\sigma s}^{\mathbf{G}}(t)$.) Thus, (4.3.18) becomes $\gamma_1/t_1 = f_1$ and $s_1 = t_1$, which yields the eigenfunction parameters associated to the TF point (t_1, f_1) as $\gamma_1 = t_1 f_1$, $s_1 = t_1$. Similarly, (4.3.19) becomes $\gamma_1/t_2 = f_2$ and $s_1\sigma = t_2$, which yields $t_2 = s_1\sigma = \sigma t_1$ and $f_2 = \gamma_1/(s_1\sigma) = f_1/\sigma$. Hence, the extended DF of $\mathbf{C}_\sigma$ is finally obtained as

$$e_{\mathbf{C}}(t_1, f_1; \sigma) \;=\; (t_2, f_2) \;=\; \Big(\sigma t_1, \frac{f_1}{\sigma}\Big)\,.$$

The extended DF of $\mathbf{G}_\gamma$ is obtained similarly. The eigenfunctions of $\mathbf{G}_\gamma$ are $u_s^{\mathbf{G}}(t) = \delta(t-s)$, with group delay $\tau_g\{u_s^{\mathbf{G}}\}(f) \equiv s$. The eigenfunctions of the dual operator $\mathbf{C}_\sigma$ are $u_\gamma^{\mathbf{C}}(t) = \frac{1}{\sqrt{t}}\, e^{j2\pi\gamma \ln(t/t_0)}$, with instantaneous frequency $f_i\{u_\gamma^{\mathbf{C}}\}(t) = \gamma/t$. (We verify that (4.3.16) is satisfied: $\big(\mathbf{G}_{\gamma'} u_\gamma^{\mathbf{C}}\big)(t) = \frac{1}{\sqrt{t}}\, e^{j2\pi\gamma \ln(t/t_0)}\, e^{j2\pi\gamma' \ln(t/t_0)} \propto u_{\gamma+\gamma'}^{\mathbf{C}}(t)$.) Thus, (4.3.18) (with the roles of instantaneous frequency and group delay as well as time and frequency interchanged) becomes $s_1 = t_1$ and $\gamma_1/t_1 = f_1$, which yields the eigenfunction parameters $s_1 = t_1$, $\gamma_1 = t_1 f_1$. Similarly, (4.3.19) (with the same interchange of roles) becomes $s_1 = t_2$ and $(\gamma_1 + \gamma)/t_2 = f_2$, whence

$t_2 = s_1 = t_1$ and $f_2 = (\gamma_1 + \gamma)/s_1 = f_1 + \gamma/t_1$. Hence, the extended DF of $\mathbf{G}_\gamma$ is obtained as

$$e_{\mathbf{G}}(t_1, f_1; \gamma) \;=\; (t_2, f_2) \;=\; \Big(t_1, f_1 + \frac{\gamma}{t_1}\Big).$$

The extended DF of $\mathbf{V}_{\sigma,\gamma} = \mathbf{G}_\gamma \mathbf{C}_\sigma$ can now be calculated by composing $e_{\mathbf{C}}(t, f; \sigma)$ and $e_{\mathbf{G}}(t, f; \gamma)$ according to (4.3.12), which yields

$$e_{\mathbf{V}}(t, f; \sigma, \gamma) \;=\; e_{\mathbf{G}}\big(e_{\mathbf{C}}(t, f; \sigma); \gamma\big) \;=\; \Big(\sigma t, \frac{f + \gamma/t}{\sigma}\Big).$$

Finally, the DF (and inverse DF) of $\mathbf{V}_{\sigma,\gamma}$ follow upon setting $t = t_0 > 0$ and $f = 0$:

$$d_{\mathbf{V}}(\sigma, \gamma) \;=\; e_{\mathbf{V}}(t_0, 0; \sigma, \gamma) \;=\; \Big(\sigma t_0, \frac{\gamma}{\sigma t_0}\Big), \qquad d_{\mathbf{V}}^{-1}(t, f) \;=\; \Big(\frac{t}{t_0}, tf\Big).$$

With the DF at our disposal, we are ready to pass from the (σ, γ) domain into the TF domain. The TF covariance properties (4.3.21) and (4.3.22) become

$$\tilde{L}_{\mathbf{V}_{\sigma',\gamma'}x}(t, f) \;=\; e^{-j2\pi(tf-\gamma')\ln\sigma'}\, \tilde{L}_x\Big(\frac{t}{\sigma'}, \sigma'\Big(f - \frac{\gamma'}{t}\Big)\Big)$$

$$\tilde{B}_{\mathbf{V}_{\sigma',\gamma'}x, \mathbf{V}_{\sigma',\gamma'}y}(t, f) \;=\; \tilde{B}_{x,y}\Big(\frac{t}{\sigma'}, \sigma'\Big(f - \frac{\gamma'}{t}\Big)\Big),$$

and the covariant linear and bilinear/quadratic TFRs are obtained from (4.3.23) and (4.3.24) as

$$\tilde{L}_x(t, f) \;=\; \sqrt{\frac{t_0}{t}} \int_0^\infty x(t')\, h^*\Big(t_0 \frac{t'}{t}\Big)\, e^{-j2\pi tf \ln(t'/t_0)}\, dt', \qquad t > 0$$

$$\tilde{B}_{x,y}(t, f) \;=\; \frac{t_0}{t} \int_0^\infty \int_0^\infty x(t_1)\, y^*(t_2)\, h^*\Big(t_0 \frac{t_1}{t}, t_0 \frac{t_2}{t}\Big)\, e^{-j2\pi tf \ln(t_1/t_2)}\, dt_1 dt_2, \qquad t > 0.$$

These TFRs are analogous to (respectively) the hyperbolic wavelet transform and the hyperbolic class introduced in [10], the difference being that in [10] the hyperbolic time-shift operator was used instead of the hyperbolic frequency-shift operator $\mathbf{G}_\gamma$.

4.3.7 Summary and Conclusions

Time-frequency representations (TFRs) that are covariant to practically important signal transformations—like time and frequency shifts, time-frequency scaling (dilation/compression), or dispersive time and frequency shifts—are of great relevance in applications. We have presented a unified and coherent *covariance theory of TFRs* that allows the systematic construction of TFRs covariant to two-parameter transformations. We note that a much more detailed and mathematically rigorous discussion with many additional references is provided in [8], where also the extension to groups not isomorphic to $(\mathbb{R}, +)$ is outlined. Furthermore, relations of covariance theory with the principle of unitary equivalence are discussed in [1] [8] (cf. also Article 4.5).

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