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Covariant Time-Frequency Analysis

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ABSTRACT We present a theory of linear and bilinear/quadratic time-frequency (TF) representations that satisfy a covariance property with respect to "TF displacement operators." These operators cause TF displacements such as (possibly dispersive) TF shifts, dilations/compressions, etc. Our covariance theory establishes a unified framework for important classes of linear TF representations (e.g., short-time Fourier transform and continuous wavelet transform) as well as bilinear TF representations (e.g., Cohen's class and affine class). It yields a theoretical basis for TF analysis and allows the systematic construction of covariant TF representations.
The covariance principle is developed both in the group domain and in the TF domain. Fundamental properties of the *displacement function* connecting these two domains and their far-reaching consequences are studied, and a method for constructing the displacement function is presented.
We also introduce important classes of operator families (modulation and warping operators; dual and affine operators), and we apply the results of the covariance theory to these operator classes. It is shown that for dual operator pairs, the characteristic function method for constructing bilinear TF representations is equivalent to the covariance method.

7.1 Introduction

Linear and bilinear/quadratic time-frequency (TF) representations are powerful tools for signal analysis [1–4]. Many important classes of TF representations can be defined by a *covariance property*. Covariance means that a TF representation reacts to certain unitary signal transformations (such as a TF shift, for example) by an associated TF coordinate transform.

In this chapter, we develop a general *covariance theory of TF analysis*, i.e., a theory of linear and bilinear TF representations based on the covariance principle. The covariance theory provides a unified framework for important linear TF representations (e.g., short-time Fourier transform and continuous wavelet transform) as well as bilinear TF representations (e.g., Cohen's class and the affine class). It is a *constructive* theory since it allows the systematic construction of covariant TF representations.

We will illustrate the covariance principle by considering some covariant TF representations of fundamental importance. Let us first define some elementary unitary signal transformations, namely, the time-shift operator $\mathbf{T}_\tau$, frequency-shift operator $\mathbf{F}_\nu$, and TF scaling operator $\mathbf{C}_\sigma$:

$$(\mathbf{T}_\tau x)(t) = x(t-\tau)\,, \qquad \tau \in \mathbb{R} \tag{7.1.1}$$

$$(\mathbf{F}_\nu x)(t) = e^{j2\pi\nu t} x(t)\,, \qquad \nu \in \mathbb{R} \tag{7.1.2}$$

$$(\mathbf{C}_\sigma x)(t) = \frac{1}{\sqrt{|\sigma|}}\, x\Big(\frac{t}{\sigma}\Big)\,, \qquad \sigma \in \mathbb{R}\backslash\{0\} \quad \text{or} \quad \sigma \in \mathbb{R}^+\,. \tag{7.1.3}$$

Using these operators, we define the TF shift operator $\mathbf{S}_{\tau,\nu}$ and the time-shift/TF scaling operator $\mathbf{R}_{\sigma,\tau}$ as

$$\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau\,, \qquad \mathbf{R}_{\sigma,\tau} = \mathbf{T}_\tau \mathbf{C}_\sigma\,.$$

We first consider some elementary families of *linear* TF representations (LTFRs), i.e., TF representations that depend on a signal under analysis, $x(t)$, in a linear manner. The classical LTFR family is the *short-time Fourier transform* [1, 3, 4]

$$\mathrm{STFT}_x(t,f) = \int_{t'} x(t')\, h^*(t'-t)\, e^{-j2\pi f t'}\, dt', \tag{7.1.4}$$

where t and f denote time and frequency, respectively, $h(t)$ is a function (known as the "window") that does not depend on $x(t)$, and integration is over the entire support of the function integrated. The short-time Fourier transform family can be shown to consist of all LTFRs L that are "covariant" (up to a phase factor) to TF shifts, in the sense that

$$L_{\mathbf{S}_{\tau,\nu}x}(t,f) = e^{-j2\pi\tau(f-\nu)}\, L_x(t-\tau, f-\nu)\,. \tag{7.1.5}$$

We can say that within the entire class of LTFRs, the short-time Fourier transform is axiomatically defined by the TF shift covariance property (7.1.5), i.e., by covariance with respect to the operator $\mathbf{S}_{\tau,\nu}$.

A second important LTFR family is the *continuous wavelet transform* [3–7]

$$\mathrm{WT}_x(t,f) = \sqrt{|f|} \int_{t'} x(t')\, h^*\big(f(t'-t)\big)\, dt', \qquad f \neq 0\,,$$

with $h(t)$ a signal-independent function (the "mother wavelet"). The wavelet transform family consists of all LTFRs L covariant to time shifts and TF scalings:

$$L_{\mathbf{R}_{\sigma,\tau}x}(t,f) = L_x\left(\frac{t-\tau}{\sigma}, \sigma f\right)\,.$$

Further LTFR families are the *hyperbolic wavelet transform* [8] and the *power wavelet transform* [9], and all other LTFR families that can be derived from the short-time Fourier transform or the wavelet transform using the principle of *unitary equivalence* [10–12]. All these LTFR families can be defined by covariance properties with respect to specific operators of the generic form $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta \mathbf{A}_\alpha$.

Let us now turn to important families of *bilinear* TF representations (BTFRs) that depend on two signals $x(t), y(t)$ in a bilinear (strictly speaking, sesquilinear) manner [13]. The classical BTFR family is *Cohen's class* (with signal-independent kernel) [1–3]

$$C_{x,y}(t,f) = \int_{t_1}\int_{t_2} x(t_1)\, y^*(t_2)\, h^*(t_1-t, t_2-t)\, e^{-j2\pi f(t_1-t_2)}\, dt_1 dt_2\,, \quad (7.1.6)$$

where $h(t_1, t_2)$ is a signal-independent function (the "kernel" function). Cohen's class is well known to consist of all BTFRs B that are covariant to TF shifts [1, 3, 14],

$$B_{\mathbf{S}_{\tau,\nu}x, \mathbf{S}_{\tau,\nu}y}(t,f) = B_{x,y}(t-\tau, f-\nu)\,. \quad (7.1.7)$$

Hence, Cohen's class is axiomatically defined by the TF shift covariance property (7.1.7).

A second important BTFR class is the *affine class* [1, 3, 15–19]

$$A_{x,y}(t,f) = |f| \int_{t_1}\int_{t_2} x(t_1)\, y^*(t_2)\, h^*\big(f\,(t_1-t), f\,(t_2-t)\big)\, dt_1 dt_2\,.$$

The affine class consists of all BTFRs B covariant to time shifts and TF scalings [15–17],

$$B_{\mathbf{R}_{\sigma,\tau}x, \mathbf{R}_{\sigma,\tau}y}(t,f) = B_{x,y}\left(\frac{t-\tau}{\sigma}, \sigma f\right)\,.$$

Further BTFR classes are the *hyperbolic class* [8, 20, 21], the *power classes* [9, 21, 22], the *exponential class* [23, 24], and all other BTFR classes that can be derived from Cohen's class or the affine class using the unitary equivalence principle [8–12, 21, 24]. All these BTFR classes can be defined by covariance properties with respect to specific operators of the generic form $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta \mathbf{A}_\alpha$.

The fact that important LTFR and BTFR classes can be defined by a covariance property has led to the development of a unified *covariance theory of TF analysis* [25–28]. The present chapter provides a discussion of this theory that takes into account later contributions [12, 28–35] and adds important new aspects. The covariance theory is based on "displacement operators" of the form (possibly up to a phase factor) $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta \mathbf{A}_\alpha$ which generalize the operators $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ and $\mathbf{R}_{\sigma,\tau} = \mathbf{T}_\tau \mathbf{C}_\sigma$. An important attribute of a displacement operator is its *displacement function* that describes its action in the TF plane.

This chapter is organized as follows. Section 7.2 discusses unitary and projective group representations that provide mathematical models for displacement operators. Two important types of unitary group representations—termed *modulation operators* and *warping operators*—are considered in some detail.

In the next two sections, two typical relations between modulation and warping operators are extended in a canonical way: Section 7.3 defines pairs of *dual operators* (generalizing the operator pair $(\mathbf{T}_\tau, \mathbf{F}_\nu)$) while Section 7.4 defines pairs of *affine operators* (generalizing the operator pair $(\mathbf{C}_\sigma, \mathbf{T}_\tau)$).

Section 7.5 introduces the concept of displacement operators $\mathbf{D}_{\alpha,\beta}$ and derives the classes of all linear and bilinear "(α, β) representations" covariant to a given displacement operator. These covariant (α, β) representations are an intermediate step in the construction of covariant TF representations.

Section 7.6 defines the *displacement function* of a displacement operator as a mapping between the (α, β) domain and the TF domain. A systematic method for constructing the displacement function is presented in Section 7.7. The displacement function is then used in Section 7.8 to convert the covariant (α, β) representations of Section 7.5 into covariant TF representations.

Section 7.9 considers the *characteristic function method* for the construction of BTFRs. It is shown that for *dual* operators the characteristic function method is equivalent to the covariance method. Finally, Section 7.10 outlines the extension of the covariance theory to general LCA groups.

7.2 Groups and Group Representations

The covariance theory is based on families of unitary operators $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$, and $\mathbf{D}_{\alpha,\beta}$ whose parameters α, β, or $\theta = (\alpha, \beta)$ belong to a *group*. These operator families have the mathematical structure of *unitary or projective group representations*. This section reviews some fundamentals and then introduces two special types of unitary group representations termed *modulation operators* and *warping operators*. Modulation and warping operators frequently occur in signal theory and will be important in later sections.

7.2.1 Groups

We shall first review some group theory fundamentals [36–39]. Let $\mathcal{G}$ be a set of elements with a binary operation $\star$ that assigns to every ordered pair of elements (g_1, g_2) with $g_1 \in \mathcal{G}$ and $g_2 \in \mathcal{G}$ a unique element $g_1 \star g_2 \in \mathcal{G}$. $(\mathcal{G}, \star)$ is called a *group* if it satisfies the following properties [36]:

1. There exists an *identity element* $g_0 \in \mathcal{G}$ such that $g \star g_0 = g_0 \star g = g$ for all $g \in \mathcal{G}$.

2. To every $g \in \mathcal{G}$ there exists an *inverse element* $g^{-1} \in \mathcal{G}$ such that $g \star g^{-1} = g^{-1} \star g = g_0$. We note that $(g_1 \star g_2)^{-1} = g_2^{-1} \star g_1^{-1}$.

3. The operation $\star$ is *associative*, i.e., $g_1 \star (g_2 \star g_3) = (g_1 \star g_2) \star g_3$, for all $g_1, g_2, g_3 \in \mathcal{G}$.

If, in addition, $g_1 \star g_2 = g_2 \star g_1$ for all $g_1, g_2 \in \mathcal{G}$, the group is called *commutative* or *abelian*.

Some elementary examples of groups are the following:

- $(\mathbb{R}, +)$, the set of all real numbers with addition, i.e., $g_1 \star g_2 = g_1 + g_2$, is a commutative group with identity element $g_0 = 0$ and inverse elements $g^{-1} = -g$.
- $(\mathbb{R}^+, \cdot)$, the set of all positive real numbers with multiplication, i.e., $g_1 \star g_2 = g_1 g_2$, is a commutative group with identity element $g_0 = 1$ and inverse elements $g^{-1} = 1/g$.
- $(\mathbb{R}^2, +)$, the set of all ordered pairs (g, h), where $g \in \mathbb{R}$ and $h \in \mathbb{R}$, together with elementwise addition, i.e., $(g_1, h_1) \star (g_2, h_2) = (g_1 + g_2,\, h_1 + h_2)$, is a commutative group. The identity element is $(0, 0)$ and the inverse elements are $(g, h)^{-1} = (-g, -h)$.
- An example of a non-commutative group is the *affine group* of ordered pairs (g, h), where $g \in \mathbb{R}^+$, $h \in \mathbb{R}$, with group operation $(g_1, h_1) \star (g_2, h_2) = (g_1 g_2,\, h_1 g_2 + h_2)$. The identity element is $(1, 0)$ and the inverse elements are $(g, h)^{-1} = (1/g, -h/g)$.

The above groups are examples of *topological* groups [36, 37, 40, 41] for which $(g_1, g_2) \mapsto g_1 \star g_2^{-1}$ is a continuous map of the product space $\mathcal{G} \times \mathcal{G}$ onto $\mathcal{G}$; here, continuity is defined with respect to some topology on $\mathcal{G}$. For instance, in our first example the mapping $(g_1, g_2) \mapsto g_1 - g_2$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ is continuous.

Let us now consider two groups $(\mathcal{G}, \star)$ and $(\mathcal{H}, \diamond)$. $(\mathcal{G}, \star)$ is said to be *isomorphic* to $(\mathcal{H}, \diamond)$ with *isomorphism* ψ if the mapping $\psi : \mathcal{G} \to \mathcal{H}$ is invertible (i.e., one-to-one and onto) and if

$$\psi(g_1 \star g_2) = \psi(g_1) \diamond \psi(g_2) \qquad \text{for all } g_1, g_2 \in \mathcal{G} .$$

If $(\mathcal{G}, \star)$ is isomorphic to $(\mathcal{H}, \diamond)$ with isomorphism ψ, then conversely $(\mathcal{H}, \diamond)$ is isomorphic to $(\mathcal{G}, \star)$ with isomorphism ψ^{-1}, i.e., $\psi^{-1}(h_1 \diamond h_2) = \psi^{-1}(h_1) \star \psi^{-1}(h_2)$ for all $h_1, h_2 \in \mathcal{H}$. For $\mathcal{G}$ and $\mathcal{H}$ topological groups, $\mathcal{G}$ is said to be *topologically isomorphic* to $\mathcal{H}$ if $(\mathcal{G}, \star)$ is isomorphic to $(\mathcal{H}, \diamond)$ and the isomorphism ψ and the inverse function ψ^{-1} are continuous. For example, $(\mathbb{R}, +)$ is topologically isomorphic to $(\mathbb{R}^+, \cdot)$ with isomorphism $\psi(g) = e^g$, $g \in \mathbb{R}$. Of special importance for our discussion are groups $(\mathcal{G}, \star)$ that are topologically isomorphic to $(\mathbb{R}, +)$. Such groups are commutative and the isomorphism $\psi_{\mathcal{G}} : \mathcal{G} \to \mathbb{R}$ satisfies $\psi_{\mathcal{G}}(g_1 \star g_2) = \psi_{\mathcal{G}}(g_1) + \psi_{\mathcal{G}}(g_2)$ for all $g_1, g_2 \in \mathcal{G}$. *Hereafter, "isomorphic" will always mean "topologically isomorphic."*

7.2.2 Unitary and Projective Group Representations

Let $(\mathcal{G}, \star)$ be a topological group with identity element g_0 and let $\mathcal{X}$ be an infinite-dimensional Hilbert space of functions (signals) $x(t)$, i.e., the space is equipped with an inner product $\langle \cdot, \cdot \rangle$ and complete. Usually, $\mathcal{X}$ will be $L^2(\Omega)$ or a subspace of $L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}$, and the inner product is of the form $\langle x, y \rangle = \int_\Omega x(t)\, y^*(t)\, dt$. A family of unitary linear operators $\{\mathbf{G}_g\}_{g \in (\mathcal{G},\star)}$ where each operator $\mathbf{G}_g$ maps $\mathcal{X}$ onto itself is called a *unitary representation of the group* $(\mathcal{G}, \star)$ *on* $\mathcal{X}$ if the following conditions are satisfied [38, 39]:

1. The map $g \mapsto \mathbf{G}_g$ is such that
$$\mathbf{G}_{g_0} = \mathbf{I} \quad \text{and} \quad \mathbf{G}_{g_2}\mathbf{G}_{g_1} = \mathbf{G}_{g_1 \star g_2} \;\text{ for all }\; g_1, g_2 \in \mathcal{G}\,, \tag{7.2.8}$$
where $\mathbf{I}$ is the identity operator on $\mathcal{X}$. Note that if $(\mathcal{G}, \star)$ is a commutative group, then $g_1 \star g_2 = g_2 \star g_1$, which implies $\mathbf{G}_{g_2}\mathbf{G}_{g_1} = \mathbf{G}_{g_1}\mathbf{G}_{g_2}$.
2. For every fixed $x \in \mathcal{X}$, the map $\mathcal{G} \to \mathcal{X}$, $g \mapsto \mathbf{G}_g\, x$ is continuous.

Elementary examples of unitary group representations are the time-shift operator $\mathbf{T}_\tau$, frequency-shift operator $\mathbf{F}_\nu$, TF scaling operator $\mathbf{C}_\sigma$, and the operators $\mathbf{R}_{\sigma,\tau} = \mathbf{T}_\tau \mathbf{C}_\sigma$ and $\mathbf{R}'_{\sigma,\tau} = \mathbf{C}_\sigma \mathbf{T}_\tau$.

Similarly, a family of unitary operators $\{\mathbf{G}_g\}_{g \in (\mathcal{G},\star)}$ defined on a Hilbert space $\mathcal{X}$ is called a *projective (or ray) representation of the group* $(\mathcal{G}, \star)$ *on* $\mathcal{X}$ if the following conditions are satisfied [42, 43]:

1. The map $g \mapsto \mathbf{G}_g$ is such that
$$\mathbf{G}_{g_0} = \mathbf{I} \quad \text{and} \quad \mathbf{G}_{g_2}\mathbf{G}_{g_1} = c(g_1, g_2)\, \mathbf{G}_{g_1 \star g_2} \;\text{ for all }\; g_1, g_2 \in \mathcal{G}\,, \tag{7.2.9}$$
with a continuous function $c : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$.
2. For every fixed $x \in \mathcal{X}$, the map $\mathcal{G} \to \mathcal{X}$, $g \mapsto \mathbf{G}_g\, x$ is continuous.

Note that if $c(g_1, g_2) \equiv 1$, then $\mathbf{G}_g$ is a unitary representation of $\mathcal{G}$. Hence, projective representations can be viewed as a generalization of unitary representations. The function $c(g_1, g_2)$ is called the *cocycle* of $\mathbf{G}_g$. From (7.2.9), it follows that the cocycle c has the following properties:

$$|c(g_1, g_2)| = 1 \quad \text{or} \quad \frac{1}{c(g_1, g_2)} = c^*(g_1, g_2) \tag{7.2.10}$$
$$c(g, g_0) = c(g_0, g) = 1$$
$$c(g_1, g_2)\, c(g_1 \star g_2, g_3) = c(g_1, g_2 \star g_3)\, c(g_2, g_3)$$
$$c(g_1, g_2^{-1})\, c(g_1 \star g_2^{-1}, g_2) = c(g_2^{-1}, g_2) \tag{7.2.11}$$
$$c(g, g^{-1}) = c(g^{-1}, g)\,, \tag{7.2.12}$$

for all $g_1, g_2, g_3 \in \mathcal{G}$. Eq. (7.2.10) shows that the cocycle is a phase factor, i.e., $c(g_1, g_2) = e^{j\phi(g_1, g_2)}$, with some phase function $\phi(g_1, g_2)$ that satisfies $\phi(g, g_0) = \phi(g_0, g) = 0 \bmod 2\pi$ and $\phi(g_1, g_2) + \phi(g_1 \star g_2, g_3) = \phi(g_1, g_2 \star g_3) + \phi(g_2, g_3) \bmod 2\pi$. Furthermore it follows from (7.2.9) and (7.2.10) with $g_1 = g$ and $g_2 = g^{-1}$ that

$$\mathbf{G}_g^{-1} = c^*(g, g^{-1})\, \mathbf{G}_{g^{-1}}. \tag{7.2.13}$$

Elementary examples of projective group representations are the TF shift operators $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ and $\mathbf{S}'_{\tau,\nu} = \mathbf{T}_\tau \mathbf{F}_\nu$ with cocycles $c\big((\tau_1, \nu_1), (\tau_2, \nu_2)\big) = e^{-j2\pi\nu_1\tau_2}$ and $c'\big((\tau_1, \nu_1), (\tau_2, \nu_2)\big) = e^{j2\pi\nu_2\tau_1}$, respectively.

A unitary or projective group representation $\mathbf{G}_g$ on a Hilbert space $\mathcal{X}$ is called *irreducible* if $\mathcal{X}$ is *minimal invariant* under $\mathbf{G}_g$, i.e., if there are no (nonzero) closed subspaces of $\mathcal{X}$ that are preserved by the action of $\mathbf{G}_g$. It is called *faithful* if $\mathbf{G}_{g_1} = \mathbf{G}_{g_2}$ implies $g_1 = g_2$ [38].

7.2.3 Modulation Operators

We shall now discuss an important type of unitary group representations, previously considered in [21, 33], that generalizes the frequency-shift operator $\mathbf{F}_\nu$ in (7.1.2).

Definition 7.1 *[33] Consider a set $\Omega \subseteq \mathbb{R}$, two groups $(\mathcal{G}, \star)$ and $(\mathcal{H}, \diamond)$ isomorphic to $(\mathbb{R}, +)$ with isomorphisms $\psi_\mathcal{G}$ and $\psi_\mathcal{H}$, respectively, and an invertible function $m(t)$ with domain Ω and range $m(\Omega) \subseteq \mathcal{G}$. Let $\tilde{m}(t) \triangleq \psi_\mathcal{G}(m(t))$. A family of linear operators $\{\mathbf{M}_h\}_{h\in(\mathcal{H},\diamond)}$ defined on $L^2(\Omega)$ as*

$$(\mathbf{M}_h x)(t) = e^{j2\pi\, \psi_\mathcal{H}(h)\, \tilde{m}(t)}\, x(t)\,, \qquad h \in (\mathcal{H}, \diamond), \quad t \in \Omega \tag{7.2.14}$$

will be called a modulation operator (family). *The function $m(t)$ will be called the* modulation function.

We note that $\tilde{m}(t) = \psi_\mathcal{G}(m(t))$ has domain $\Omega \subseteq \mathbb{R}$ and its range is $\subseteq \mathbb{R}$. It is easily checked that the modulation operator $\{\mathbf{M}_h\}_{h\in\mathcal{H}}$ is a unitary representation of the group $(\mathcal{H}, \diamond)$; in particular, $\mathbf{M}_h$ is unitary and satisfies $\mathbf{M}_{h_2}\mathbf{M}_{h_1} = \mathbf{M}_{h_1}\mathbf{M}_{h_2} = \mathbf{M}_{h_1 \diamond h_2}$ for all $h_1, h_2 \in \mathcal{H}$. Two examples of modulation operators are the frequency-shift operator $\mathbf{F}_\nu$, with $\nu \in (\mathbb{R}, +)$ and $x \in L^2(\mathbb{R})$, and the "hyperbolic frequency-shift operator" defined as $(\mathbf{M}_h x)(t) = e^{j2\pi h \ln t}\, x(t)$ with $h \in (\mathbb{R}, +)$, $t \in \mathbb{R}^+$, and $x \in L^2(\mathbb{R}^+)$. Furthermore, a dual (nonequivalent) definition of modulation operators can be given in the frequency domain; examples of such "frequency-domain modulation operators" are the time-shift operator $\mathbf{T}_\tau$ in (7.1.1) and the hyperbolic time-shift operator defined in [8].

It can be shown that the (generalized [44–46]) eigenfunctions and eigenvalues of $\mathbf{M}_h$ are given by

$$u_g^{\mathbf{M}}(t) \;=\; r(t)\, \delta\big(\tilde{m}(t) - \psi_\mathcal{G}(g)\big), \qquad t \in \Omega, \;\; g \in (\mathcal{G}, \star) \tag{7.2.15}$$

$$\lambda^{\mathbf{M}}_{h,g} = e^{j2\pi\,\psi_{\mathcal{H}}(h)\,\psi_{\mathcal{G}}(g)}, \qquad h \in (\mathcal{H}, \diamond), \;\; g \in (\mathcal{G}, \star)\,,$$

respectively, with $\delta(\cdot)$ the Dirac delta function. Assuming $\tilde{m}(t)$ to be differentiable and $m(\Omega) = \mathcal{G}$, the choice $r(t) = \sqrt{|\tilde{m}'(t)|}$ (where $'$ denotes differentiation) guarantees that the eigenfunctions $\{u^{\mathbf{M}}_g(t)\}_{g\in\mathcal{G}}$ are complete and orthonormal (in a generalized functions sense [44–46]) in $L^2(\Omega)$. Any modulation operator $\mathbf{M}_h$ is unitarily equivalent (up to the parameter transformation $\nu = \psi_{\mathcal{H}}(h)$) to the frequency-shift operator $\mathbf{F}_\nu$,

$$\mathbf{M}_h = \mathbf{U}\,\mathbf{F}_{\psi_{\mathcal{H}}(h)}\mathbf{U}^{-1}, \tag{7.2.16}$$

with $\mathbf{U} : L^2(\mathbb{R}) \to L^2(\Omega)$, $(\mathbf{U}x)(t) = \sqrt{|\tilde{m}'(t)|}\, x\big(\tilde{m}(t)\big)$.

7.2.4 Warping Operators

A second important type of unitary group representations, previously introduced in [47], generalizes the time-shift operator $\mathbf{T}_\tau$ in (7.1.1) and the TF scaling operator $\mathbf{C}_\sigma$ in (7.1.3):

Definition 7.2 *[33] Consider a set $\Theta \subseteq \mathbb{R}$ and a group $(\mathcal{G}, \star)$ isomorphic to $(\mathbb{R}, +)$ with isomorphism $\psi_{\mathcal{G}}$. Let $w_g(t)$ with $g \in (\mathcal{G}, \star)$ and $t \in \Theta$ be an indexed function with the following properties:*

- *For fixed $g \in (\mathcal{G}, \star)$, $w_g(t)$ is an invertible, continuously differentiable function that maps Θ onto Θ with nonzero derivative (i.e., a* diffeomorphism*) and that satisfies the composition property*

$$w_{g_1}\big(w_{g_2}(t)\big) = w_{g_1 \star g_2}(t)\,. \tag{7.2.17}$$

  (Note that this implies $w_{g_0}(t) = t$.)

- *For fixed t, the map $g \to w_g(t)$ is continuous.*

Then, a family of linear operators $\{\mathbf{W}_g\}_{g\in(\mathcal{G},\star)}$ defined on $L^2(\Theta)$ as

$$(\mathbf{W}_g\, x)(t) = \sqrt{|w'_g(t)|}\; x\big(w_g(t)\big)\,, \qquad g \in (\mathcal{G}, \star), \quad t \in \Theta \tag{7.2.18}$$

will be called a warping operator (family). *The function family $w_g(t)$ will be called the* warping function.

The warping operator $\{\mathbf{W}_g\}_{g\in\mathcal{G}}$ is a unitary representation of the group $(\mathcal{G}, \star)$. Elementary examples are the operators $\mathbf{T}_\tau$, with $\tau \in (\mathbb{R}, +)$ and $x \in L^2(\mathbb{R})$, and $\mathbf{C}_\sigma$, with $\sigma \in (\mathbb{R}^+, \cdot)$ and $x \in L^2(\mathbb{R}^+)$. Furthermore, a dual (nonequivalent) definition of warping operators can be given in the frequency domain.

The next theorem states a fundamental relation between warping and modulation functions. For the special case $(\mathcal{G}, \star) = (\mathbb{R}, +)$, a related result has been proved in [48] and (using a different argument) in [47].

Theorem 7.3 *[33] Let $\Omega \subseteq \mathbb{R}$ and let $(\mathcal{G}, \star)$ be a group isomorphic to $(\mathbb{R}, +)$ with isomorphism $\psi_{\mathcal{G}}$. If $m(t)$ is a modulation function on Ω with range $m(\Omega) = \mathcal{G}$, i.e., $m(t)$ is an invertible mapping of Ω onto $\mathcal{G}$, then*

$$w_g(t) \triangleq m^{-1}\big(m(t) \star g^{-1}\big) = \tilde{m}^{-1}\big(\tilde{m}(t) - \psi_{\mathcal{G}}(g)\big), \quad t \in \Omega, \quad g \in (\mathcal{G}, \star),$$

with $\tilde{m}(t) = \psi_{\mathcal{G}}(m(t))$, is invertible with range $w_g(\Omega) = \Omega$ and it satisfies the composition property (7.2.17). (Note that $m^{-1}(\cdot)$ denotes the inverse function associated to $m(t)$ whereas g^{-1} denotes the group inverse element associated to g. Furthermore note that $m(\Omega) = \mathcal{G}$ is equivalent to $\tilde{m}(t) = \mathbb{R}$.) If $\tilde{m}(t) = \psi_{\mathcal{G}}(m(t))$ is a diffeomorphism, then so is $w_g(t)$ (with g fixed) and the map $g \mapsto w_g(t)$ is continuous for fixed $t \in \Omega$, i.e., $w_g(t)$ is a warping function on Ω.

Conversely, consider a group $(\mathcal{G}, \star)$ isomorphic to $(\mathbb{R}, +)$ with isomorphism $\psi_{\mathcal{G}}$, and a function $w_g(t)$ with $g \in (\mathcal{G}, \star)$ and $t \in \Theta \subseteq \mathbb{R}$ that satisfies (7.2.17). If $f(g) \triangleq w_{g^{-1}}(t_0)$ ($g \in \mathcal{G}$; $t_0 \in \Theta$ arbitrary but fixed) is invertible with range $f(\mathcal{G}) = \Omega \subseteq \Theta$, then $w_g(t) = f\big(f^{-1}(t) \star g^{-1}\big)$ for $t \in \Omega$, $g \in (\mathcal{G}, \star)$, or equivalently, setting $m(t) \triangleq f^{-1}(t)$,

$$w_g(t) = m^{-1}\big(m(t) \star g^{-1}\big), \qquad t \in \Omega, \quad g \in (\mathcal{G}, \star). \tag{7.2.19}$$

Here, $m : \Omega \to \mathcal{G}$ is invertible and unique up to group translations, i.e., $m_1^{-1}\big(m_1(t) \star g^{-1}\big) \equiv m_2^{-1}\big(m_2(t) \star g^{-1}\big)$ implies $m_2(t) \equiv m_1(t) \star \kappa$ with some $\kappa \in (\mathcal{G}, \star)$. Finally, there is $w_g(\Omega) = \Omega$.

Proof. Since $m(t)$ is invertible so is $m^{-1}(g)$, and hence $w_g(t) = m^{-1}\big(m(t) \star g^{-1}\big)$ is invertible for fixed g. From $m(\Omega) = \mathcal{G}$ and $m^{-1}(\mathcal{G}) = \Omega$, it follows that $w_g(\Omega) = \Omega$. It is easily verified that $w_g(t) = m^{-1}\big(m(t) \star g^{-1}\big)$ satisfies the composition property (7.2.17). Finally, if $\tilde{m}$ is a diffeomorphism, the inverse function $\tilde{m}^{-1}$ is also a diffeomorphism, from which it follows that $w_g(t) = \tilde{m}^{-1}\big(\tilde{m}(t) - \psi_{\mathcal{G}}(g)\big)$ is a diffeomorphism for fixed g. Since both $\tilde{m}^{-1}$ and $\psi_{\mathcal{G}}$ are continuous, $w_g(t)$ is continuous with respect to g.

Our proof of the converse statement generalizes a proof in [48]. Let $f(g) \triangleq w_{g^{-1}}(t_0)$ with fixed $t_0 \in \Theta$. Then $w_g\big(f(g_1)\big) = w_g\big(w_{g_1^{-1}}(t_0)\big) = w_{g \star g_1^{-1}}(t_0) = f(g_1 \star g^{-1})$. Setting $f(g_1) = t \in \Omega \subseteq \Theta$, we obtain

$$w_g(t) = f\big(f^{-1}(t) \star g^{-1}\big). \tag{7.2.20}$$

Since by assumption $f(g)$ is invertible on $f(\mathcal{G}) = \Omega \subseteq \Theta$, Eq. (7.2.20) holds for all $t \in \Omega \subseteq \Theta$ and $g \in \mathcal{G}$. Furthermore, (7.2.20) implies that the range of $w_g(t)$ is $w_g(\Omega) = f(\mathcal{G}) = \Omega$. To show that f is unique up to group translations, assume there is a function $\tilde{f}$ such that $w_g(t) = \tilde{f}\big(\tilde{f}^{-1}(t) \star g^{-1}\big)$. Then $w_{g_1}\big(\tilde{f}(g_2)\big) = \tilde{f}(g_2 \star g_1^{-1})$, but at the same time $w_{g_1}\big(\tilde{f}(g_2)\big) = f\big(f^{-1}\big(\tilde{f}(g_2)\big) \star g_1^{-1}\big)$. Hence, setting $g_1^{-1} = g$ and $g_2 = g_0$, we obtain $\tilde{f}(g \star g_0) = f\big(f^{-1}\big(\tilde{f}(g_0)\big) \star g\big)$, i.e., $\tilde{f}(g) = f(\kappa \star g)$ with $\kappa = f^{-1}\big(\tilde{f}(g_0)\big) \in \mathcal{G}$ fixed. $\square$

Theorem 7.3 states that, under appropriate assumptions (in particular, $m(\Omega) = \mathcal{G}$ and not just $m(\Omega) \subseteq \mathcal{G}$ as required in Definition 7.1), every modulation function $m(t)$ induces an associated warping function $w_g(t) = m^{-1}\big(m(t) \star g^{-1}\big)$, and conversely, again under appropriate assumptions, every warping function $w_g(t)$ on a set Θ is generated—at least on a suitable subset $\Omega \subseteq \Theta$—by an associated modulation function $m(t)$. If $\Omega \subset \Theta$, i.e., $\Omega \neq \Theta$, then Θ can be partitioned into disjoint subsets Ω_i with associated modulation functions $m_i(t)$, $t \in \Omega_i$, such that $w_g(t) = m_i^{-1}\big(m_i(t) \star g^{-1}\big)$ for $t \in \Omega_i$ [47, 49].

This relation between modulation and warping functions immediately translates into a relation between modulation and warping operators:

Corollary 7.4 *Let $(\mathcal{G}, \star)$ and $(\mathcal{H}, \diamond)$ be groups isomorphic to $(\mathbb{R}, +)$ with isomorphisms $\psi_\mathcal{G}$ and $\psi_\mathcal{H}$. Let $\{\mathbf{M}_h\}_{h\in\mathcal{H}}$ be a modulation operator defined on the Hilbert space $\mathcal{X} = L^2(\Omega)$, based on an invertible modulation function $m : \Omega \to \mathcal{G}$ with $m(\Omega) = \mathcal{G}$. Then, if $\tilde{m}(t) = \psi_\mathcal{G}(m(t))$ is a diffeomorphism, there exists a warping operator $\{\mathbf{W}_g\}_{g\in(\mathcal{G},\star)}$ on $\mathcal{X}$ with warping function $w_g(t) = m^{-1}(m(t) \star g^{-1})$.*

Conversely, let $\{\mathbf{W}_g\}_{g\in(\mathcal{G},\star)}$ be a warping operator on $\mathcal{X} = L^2(\Theta)$ whose warping function $w_g(t)$ satisfies the condition of the converse part of Theorem 7.3. Then there exist a set $\Omega \subseteq \Theta$ and a modulation operator $\mathbf{M}_h$ on $\mathcal{X}_\Omega = L^2(\Omega) \subseteq \mathcal{X}$ whose modulation function $m(t)$ is associated to $w_g(t)$ according to $w_g(t) = m^{-1}(m(t) \star g^{-1})$. This modulation operator is a unitary representation of any group $(\mathcal{H}, \diamond)$ that is isomorphic to $(\mathbb{R}, +)$ via some isomorphism $\psi_\mathcal{H}$; it is uniquely defined up to a phase factor of the form $e^{j2\pi\psi_\mathcal{H}(\kappa)\psi_\mathcal{G}(\gamma)}$, where $\gamma \in \mathcal{G}, \kappa \in \mathcal{H}$.

We shall illustrate the relation between modulation and warping operators by discussing two examples. First, consider the modulation function $m(t) = t$ on $\Omega = \mathbb{R}$. Since $m(\Omega) = \mathbb{R}$, we must choose $(\mathcal{G}, \star) = (\mathbb{R}, +)$, whence $\psi_\mathcal{G}(g) = g$ and $\tilde{m}(t) \equiv \psi_\mathcal{G}(m(t)) = t$. Setting $(\mathcal{H}, \diamond) = (\mathbb{R}, +)$, whence $\psi_\mathcal{H}(h) = h$, the modulation operator is obtained as the frequency shift operator, $(\mathbf{M}_h\, x)(t) = e^{j2\pi ht}\, x(t)$ with $t \in \Omega = \mathbb{R}$, $h \in (\mathcal{H}, \diamond) = (\mathbb{R}, +)$. The associated warping function is $w_g(t) \equiv m^{-1}\big(m(t) \star g^{-1}\big) = t - g$ with $t \in \Omega = \mathbb{R}$, $g \in (\mathcal{G}, \star) = (\mathbb{R}, +)$, so that the associated warping operator is the time shift operator, $(\mathbf{W}_g\, x)(t) = x(t - g)$.

Next, consider $m(t) = t$ on $\Omega = \mathbb{R}^+$. Here, $m(\Omega) = \mathbb{R}^+$ so that we must choose $(\mathcal{G}, \star) = (\mathbb{R}^+, \cdot)$, isomorphic to $(\mathbb{R}, +)$ by $\psi_\mathcal{G}(g) = \ln g$. Hence, $\tilde{m}(t) \equiv \psi_\mathcal{G}(m(t)) = \ln t$. Setting $(\mathcal{H}, \diamond) = (\mathbb{R}, +)$, whence $\psi_\mathcal{H}(h) = h$, the modulation operator is obtained as $(\mathbf{M}_h x)(t) = e^{j2\pi h \ln t}\, x(t)$, $t \in \mathbb{R}^+$, $h \in (\mathbb{R}, +)$. The associated warping function and warping operator are $w_g(t) \equiv m^{-1}\big(m(t) \star g^{-1}\big) = t/g$ and $(\mathbf{W}_g\, x)(t) = \sqrt{1/g}\; x(t/g)$, respectively, with $t \in \mathbb{R}^+$, $g \in (\mathbb{R}^+, \cdot)$. These and some more examples are listed in Table 7.1.

Using the modulation function $m(t)$ (or, equivalently, $\tilde{m}(t) = \psi_\mathcal{G}(m(t))$ associated to $w_g(t)$ according to (7.2.19), we can give the following expres-

$m(t)$	Ω or Ω_i	$(\mathcal{G},\star)$	$w_g(t)$	Θ
t	$\mathbb{R}$	$(\mathbb{R},+)$	$t-g$	$\mathbb{R}$
$\lvert t\rvert$	$\mathbb{R}^+$, $\mathbb{R}^-$	$(\mathbb{R}^+,\cdot)$	t/g	$\mathbb{R}$
$\lvert t\rvert^\kappa$ $(\kappa>0)$	$\mathbb{R}^+$, $\mathbb{R}^-$	$(\mathbb{R}^+,\cdot)$	$g^{-1/\kappa}\,t$	$\mathbb{R}$
$\mathrm{sgn}(t)\lvert t\rvert^\kappa$ $(\kappa>0)$	$\mathbb{R}$	$(\mathbb{R},+)$	$\mathrm{sgn}\big(\mathrm{sgn}(t)\lvert t\rvert^\kappa-g\big)\cdot\big\lvert\mathrm{sgn}(t)\lvert t\rvert^\kappa-g\big\rvert^{1/\kappa}$	$\mathbb{R}$
$\ln\lvert t\rvert$	$\mathbb{R}^+$, $\mathbb{R}^-$	$(\mathbb{R},+)$	$e^{-g}t$	$\mathbb{R}$
$\lvert\ln t\rvert$	$(0,1)$, $(1,\infty)$	$(\mathbb{R}^+,\cdot)$	$t^{1/g}$	$\mathbb{R}^+$
e^t	$\mathbb{R}$	$(\mathbb{R}^+,\cdot)$	$t-\ln g$	$\mathbb{R}$

TABLE 7.1. *Some modulation functions and associated warping functions. Note that* $m(\Omega)=\mathcal{G}$ *and* $g\in(\mathcal{G},\star)$.

sions for the generalized eigenfunctions and eigenvalues of $\mathbf{W}_g$:

$$u_h^{\mathbf{W}}(t) = \sqrt{|\tilde{m}'(t)|}\, e^{j2\pi\,\psi_{\mathcal{H}}(h)\,\tilde{m}(t)}, \qquad t\in\Omega,\;\; h\in(\mathcal{H},\diamond) \tag{7.2.21}$$
$$\lambda_{g,h}^{\mathbf{W}} = e^{-j2\pi\,\psi_{\mathcal{G}}(g)\,\psi_{\mathcal{H}}(h)}, \qquad g\in(\mathcal{G},\star),\;\; h\in(\mathcal{H},\diamond)\,,$$

where $(\mathcal{H},\diamond)$ is any group isomorphic to $(\mathbb{R},+)$ with some isomorphism $\psi_{\mathcal{H}}$. The eigenfunctions $u_h^{\mathbf{W}}(t)$ are complete and orthonormal in $L^2(\Omega)$ in a generalized functions sense. Furthermore, it can be shown that on $L^2(\Omega)$, $\mathbf{W}_g$ is unitarily equivalent to $\mathbf{T}_\tau$ via the same operator as in (7.2.16), i.e.,

$$\mathbf{W}_g = \mathbf{U}\,\mathbf{T}_{\psi_{\mathcal{G}}(g)}\mathbf{U}^{-1}, \tag{7.2.22}$$

with $\mathbf{U}: L^2(\mathbb{R})\to L^2(\Omega)$, $(\mathbf{U}x)(t)=\sqrt{|\tilde{m}'(t)|}\,x\big(\tilde{m}(t)\big)$.

7.3 Dual Operators

The time-shift operator $\mathbf{T}_\tau$ and the frequency-shift operator $\mathbf{F}_\nu$ are unitary representations of the group $(\mathbb{R},+)$ on the Hilbert space $L^2(\mathbb{R})$. They satisfy the commutation relation

$$\mathbf{F}_\nu\mathbf{T}_\tau = e^{j2\pi\tau\nu}\,\mathbf{T}_\tau\mathbf{F}_\nu$$

and several other interesting relations. The strong relationship existing between $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$ is generalized by the concept of *dual operators* (originally termed *conjugate operators* in [26, 30]). In this section, we define dual operators, study some of their properties, and show that associated modulation and warping operators are dual.

Definition 7.5 *[29, 30] Let* $(\mathcal{G},\star)$ *and* $(\mathcal{H},\diamond)$ *be two groups isomorphic to* $(\mathbb{R},+)$ *with isomorphisms* $\psi_{\mathcal{G}}$ *and* $\psi_{\mathcal{H}}$*, respectively, and consider two unitary representations* $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ *and* $\{\mathbf{H}_h\}_{h\in\mathcal{H}}$ *of* $(\mathcal{G},\star)$ *and* $(\mathcal{H},\diamond)$*, respectively, on a minimal invariant Hilbert space* $\mathcal{X}$ *(i.e.,* $\mathcal{X}$ *is the smallest* *nonzero space that is invariant under both* $\mathbf{G}_g$ *and* $\mathbf{H}_h$*). Then,* $\{\mathbf{H}_h\}_{h\in\mathcal{H}}$ *is called* dual to $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ *if*

$$\mathbf{H}_h\mathbf{G}_g = e^{j2\pi\,\psi_{\mathcal{G}}(g)\,\psi_{\mathcal{H}}(h)}\,\mathbf{G}_g\mathbf{H}_h\,, \quad \textit{for all } g\in(\mathcal{G},\star),\ \ h\in(\mathcal{H},\diamond)\,. \tag{7.3.23}$$

Note that when $\mathbf{H}_h$ is dual to $\mathbf{G}_g$, then at the same time $\mathbf{G}_{g^{-1}}$ is dual to $\mathbf{H}_h$ and $\mathbf{G}_g$ is dual to $\mathbf{H}_{h^{-1}}$. The time-shift operator $\mathbf{T}_\tau$ and the frequency-shift operator $\mathbf{F}_\nu$ are the most elementary example of dual operators ($\mathbf{F}_\nu$ is dual to $\mathbf{T}_\tau$). Further examples will be considered in Subsection 7.3.2.

7.3.1 Properties of Dual Operators

The following theorem states a relation of dual operators with the two-parameter group $(\mathbb{R}^2,+)$. As we will see later, this result implies that we can use the composition $\mathbf{D}_{g,h} = \mathbf{H}_h\mathbf{G}_g$ or $\mathbf{D}'_{g,h} = \mathbf{G}_g\mathbf{H}_h$ of dual operators $\mathbf{G}_g$, $\mathbf{H}_h$ as "displacement operators" for constructing covariant TF signal representations.

Theorem 7.6 *Let* $(\mathcal{G},\star)$ *and* $(\mathcal{H},\diamond)$ *be isomorphic to* $(\mathbb{R},+)$ *with isomorphisms* $\psi_{\mathcal{G}}$ *and* $\psi_{\mathcal{H}}$*, respectively. Let* $\{\mathbf{H}_h\}_{h\in\mathcal{H}}$ *be dual to* $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ *on a minimal invariant Hilbert space* $\mathcal{X}$*. Then the unitary operator* $\mathbf{D}_{g,h} \triangleq \mathbf{H}_h\mathbf{G}_g$ *is an irreducible and faithful projective representation of the group* $(\mathcal{G},\star)\times(\mathcal{H},\diamond)$ *that is isomorphic to* $(\mathbb{R}^2,+)$ *with isomorphism* $\psi_{\mathcal{G}\times\mathcal{H}}(g,h) = \big(\psi_{\mathcal{G}}(g),\psi_{\mathcal{H}}(h)\big)$*, on* $\mathcal{X}$*, i.e.,*

$$\mathbf{D}_{g_0,h_0} = \mathbf{I}\,, \qquad \mathbf{D}_{g_2,h_2}\mathbf{D}_{g_1,h_1} = e^{-j2\pi\,\psi_{\mathcal{G}}(g_2)\,\psi_{\mathcal{H}}(h_1)}\,\mathbf{D}_{g_1\star g_2,\,h_1\diamond h_2}\,. \tag{7.3.24}$$

Similarly, also $\mathbf{D}'_{g,h} \triangleq \mathbf{G}_g\mathbf{H}_h$ *is an irreducible and faithful projective representation of* $(\mathcal{G},\star)\times(\mathcal{H},\diamond)$ *on* $\mathcal{X}$*,*

$$\mathbf{D}'_{g_0,h_0} = \mathbf{I}\,, \qquad \mathbf{D}'_{g_2,h_2}\mathbf{D}'_{g_1,h_1} = e^{j2\pi\,\psi_{\mathcal{G}}(g_1)\,\psi_{\mathcal{H}}(h_2)}\,\mathbf{D}'_{g_1\star g_2,\,h_1\diamond h_2}\,.$$

Proof. Clearly, $\mathbf{D}_{g_0,h_0} = \mathbf{H}_{h_0}\mathbf{G}_{g_0} = \mathbf{I}$. Furthermore, due to (7.3.23) and (7.2.8) there is

$$\begin{aligned}\mathbf{D}_{g_2,h_2}\mathbf{D}_{g_1,h_1} &= \mathbf{H}_{h_2}\mathbf{G}_{g_2}\mathbf{H}_{h_1}\mathbf{G}_{g_1} \;=\; e^{-j2\pi\,\psi_{\mathcal{G}}(g_2)\,\psi_{\mathcal{H}}(h_1)}\mathbf{H}_{h_2}\mathbf{H}_{h_1}\mathbf{G}_{g_2}\mathbf{G}_{g_1}\\ &= e^{-j2\pi\,\psi_{\mathcal{G}}(g_2)\,\psi_{\mathcal{H}}(h_1)}\mathbf{H}_{h_1\diamond h_2}\mathbf{G}_{g_1\star g_2} \;=\; e^{-j2\pi\,\psi_{\mathcal{G}}(g_2)\,\psi_{\mathcal{H}}(h_1)}\mathbf{D}_{g_1\star g_2,\,h_1\diamond h_2}\,,\end{aligned}$$

which is the second equation in (7.3.24). Hence, $\mathbf{D}_{g,h}$ is a projective representation of the group $(\mathcal{G}\times\mathcal{H},\circ) = (\mathcal{G},\star)\times(\mathcal{H},\diamond)$ with operation $(g_1,h_1)\circ(g_2,h_2) = (g_1\star g_2, h_1\diamond h_2)$. This group is easily seen to be isomorphic to $(\mathbb{R}^2,+)$ with isomorphism $\psi_{\mathcal{G}\times\mathcal{H}}(g,h) = (\psi_{\mathcal{G}}(g),\psi_{\mathcal{H}}(h))$. Finally, since $\mathcal{X}$ is minimal invariant under $\mathbf{G}_g$ and $\mathbf{H}_h$, it is also minimal invariant under the composition $\mathbf{D}_{g,h} = \mathbf{H}_h\mathbf{G}_g$; hence, $\mathbf{D}_{g,h}$ is irreducible. The faithfulness of $\mathbf{D}_{g,h}$ follows from Theorem 7.7. The proof for $\mathbf{D}'_{g,h}$ is analogous. □

The next theorem, proved in [50–52] (see also [29]), states a relation of dual operators to $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$.

Theorem 7.7 *Let $(\mathcal{G}, \star)$ and $(\mathcal{H}, \diamond)$ be isomorphic to $(\mathbb{R}, +)$ with isomorphisms $\psi_\mathcal{G}$ and $\psi_\mathcal{H}$, respectively, and let $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ and $\{\mathbf{H}_h\}_{h\in\mathcal{H}}$ be two unitary representations on a minimal invariant Hilbert space $\mathcal{X}$. Then, $\{\mathbf{H}_h\}_{h\in\mathcal{H}}$ is dual to $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ if and only if $\mathbf{G}_g$ and $\mathbf{H}_h$ are unitarily equivalent, up to parameter transformations by $\psi_\mathcal{G}$ and $\psi_\mathcal{H}$, to (respectively) $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$ on $L^2(\mathbb{R})$, i.e.,*

$$\mathbf{G}_g = \mathbf{U}\mathbf{T}_{\psi_\mathcal{G}(g)}\mathbf{U}^{-1} \quad \textit{and} \quad \mathbf{H}_h = \mathbf{U}\mathbf{F}_{\psi_\mathcal{H}(h)}\mathbf{U}^{-1}\,, \tag{7.3.25}$$

where $\mathbf{U} : L^2(\mathbb{R}) \to \mathcal{X}$ is an invertible and isometric (i.e., inner product preserving) operator.

For a Hilbert space $\mathcal{X} = L^2(\Omega)$, the operators $\mathbf{U}$ and $\mathbf{U}^{-1}$ are explicitly given in terms of the (generalized) eigenfunctions of $\mathbf{G}_g$ [29, 49]. The eigenfunctions $u_h^{\mathbf{G}}(t)$ of $\mathbf{G}_g$ do not depend on g, and they are parameterized by a parameter h that belongs to some group isomorphic to $(\mathbb{R}, +)$. This parameterization can always be chosen such that this group is $\mathcal{H}$. The system of eigenfunctions $\{u_h^{\mathbf{G}}(t)\}_{h\in\mathcal{H}}$ is complete and orthogonal in $\mathcal{X}$ in a generalized functions sense. For $\mathcal{X} = L^2(\Omega)$, we then have for $\mathbf{U}$ and $\mathbf{U}^{-1}$

$$\mathbf{U} = \widetilde{\mathbf{U}}\mathbb{F}^{-1} \quad \text{with} \quad (\widetilde{\mathbf{U}}y)(t) = \int_\mathbb{R} y(r)\, u^{\mathbf{G}}_{\psi_\mathcal{H}^{-1}(r)}(t)\, dr\,, \quad t \in \Omega \tag{7.3.26}$$

$$\begin{aligned}\mathbf{U}^{-1} = \mathbb{F}\widetilde{\mathbf{U}}^{-1} \quad \text{with} \quad (\widetilde{\mathbf{U}}^{-1}x)(r) &= \left\langle x, u^{\mathbf{G}}_{\psi_\mathcal{H}^{-1}(r)}\right\rangle \\ &= \int_\Omega x(t)\, u^{\mathbf{G}*}_{\psi_\mathcal{H}^{-1}(r)}(t)\, dt\,, \quad r \in \mathbb{R},\end{aligned}$$

where $\mathbb{F}$ denotes Fourier transform. We note that the transformation $\widetilde{\mathbf{U}}^{-1}$ is a reparameterized version of the so-called $\mathbf{G}_g$*-Fourier transform* [29, 45].

These results allow the construction of the dual operator $\mathbf{H}_h$ for a given operator $\mathbf{G}_g$: Let $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ be a unitary representation of $(\mathcal{G}, \star)$ on $\mathcal{X}$ that is unitarily equivalent to $\mathbf{T}_\tau$ as in (7.3.25). Then the eigenfunctions of $\mathbf{G}_g$ define the operator $\mathbf{U}$ according to (7.3.26), and the operator $\mathbf{H}_h$ dual to $\mathbf{G}_g$ is obtained from the second equation in (7.3.25).

For later use, we finally note two eigenfunction relations of dual operators. Let $\{\mathbf{H}_h\}_{h\in(\mathcal{H},\diamond)}$ be dual to $\{\mathbf{G}_g\}_{g\in(\mathcal{G},\star)}$, and let $\{u_h^{\mathbf{G}}(t)\}_{h\in(\mathcal{H},\diamond)}$ and $\{u_g^{\mathbf{H}}(t)\}_{g\in(\mathcal{G},\star)}$ denote the eigenfunctions of $\mathbf{G}_g$ and $\mathbf{H}_h$, respectively. Then it can be shown [26, 30] that, assuming suitable parameterization of the eigenfunctions,

$$\big(\mathbf{G}_{g_1}u_g^{\mathbf{H}}\big)(t) = u^{\mathbf{H}}_{g\star g_1}(t)\,, \qquad \big(\mathbf{H}_{h_1}u_h^{\mathbf{G}}\big)(t) = u^{\mathbf{G}}_{h\diamond h_1}(t)\,. \tag{7.3.27}$$

That is, $\mathbf{G}_g$ maps an eigenfunction $u_g^{\mathbf{H}}(t)$ of the dual operator $\mathbf{H}_h$ again onto an eigenfunction of $\mathbf{H}_h$ with the eigenfunction parameter g translated by g_1, and similarly for $\mathbf{H}_h$.

$m(t)$	Ω	$(\mathcal{H},\diamond)$	$(\mathcal{G},\star)$	$w_g(t)$	$(\mathbf{M}_h x)(t)$	$(\mathbf{W}_g x)(t)$
t	$\mathbb{R}$	$(\mathbb{R},+)$	$(\mathbb{R},+)$	$t-g$	$e^{j2\pi ht}\,x(t)$	$x(t-g)$
t	$\mathbb{R}^+$	$(\mathbb{R},+)$	$(\mathbb{R}^+,\cdot)$	t/g	$e^{j2\pi h\ln t}\,x(t)$	$\frac{1}{\sqrt{g}}\,x\big(\frac{t}{g}\big)$
$\ln t$	$(1,\infty)$	$(\mathbb{R},+)$	$(\mathbb{R}^+,\cdot)$	$t^{1/g}$	$e^{j2\pi h\ln(\ln t)}\,x(t)$	$\sqrt{\frac{1}{g}\,t^{1/g\,-\,1}}\,x\big(t^{1/g}\big)$
e^t	$\mathbb{R}$	$(\mathbb{R},+)$	$(\mathbb{R}^+,\cdot)$	$t-\ln g$	$e^{j2\pi ht}\,x(t)$	$x(t-\ln g)$
$1/t$	$\mathbb{R}^+$	$(\mathbb{R},+)$	$(\mathbb{R}^+,\cdot)$	gt	$e^{-j2\pi h\ln t}\,x(t)$	$\sqrt{g}\,x(gt)$

TABLE 7.2. *Some dual modulation and warping operators along with the underlying modulation and warping functions. We note that* $m(\Omega)=\mathcal{G}$, $g\in(\mathcal{G},\star)$, *and* $h\in(\mathcal{H},\diamond)$.

7.3.2 *Modulation and Warping Operators as Dual Operators*

Let $(\mathcal{G},\star)$ and $(\mathcal{H},\diamond)$ be groups isomorphic to $(\mathbb{R},+)$ with isomorphisms $\psi_{\mathcal{G}}$ and $\psi_{\mathcal{H}}$, respectively. Consider the modulation operator $\{\mathbf{M}_h\}_{h\in(\mathcal{H},\diamond)}$ generated by some modulation function $m(t)$ defined on some set Ω, with $m(\Omega)=\mathcal{G}$, and the warping operator $\{\mathbf{W}_g\}_{g\in(\mathcal{G},\star)}$ that is *associated* to $\{\mathbf{M}_h\}_{h\in(\mathcal{H},\diamond)}$ in the sense of Corollary 7.4, i.e., the underlying warping function $w_g(t)$ is induced by $m(t)$ according to $w_g(t)=m^{-1}\big(m(t)\star g^{-1}\big)$. Note that $\mathbf{M}_h$ and $\mathbf{W}_g$ are defined on the same Hilbert space $\mathcal{X}=L^2(\Omega)$.

According to (7.2.16) and (7.2.22), $\mathbf{M}_h$ and $\mathbf{W}_g$ are unitarily equivalent to $\mathbf{F}_\nu$ and $\mathbf{T}_\tau$, respectively, i.e., $\mathbf{M}_h=\mathbf{U}\,\mathbf{F}_{\psi_{\mathcal{H}}(h)}\mathbf{U}^{-1}$ and $\mathbf{W}_g=\mathbf{U}\,\mathbf{T}_{\psi_{\mathcal{G}}(g)}\mathbf{U}^{-1}$, with $(\mathbf{U}x)(t)=\sqrt{|\tilde{m}'(t)|}\,x\big(\tilde{m}(t)\big)$. With Theorem 7.7, we then obtain the important result that *associated modulation and warping operators are dual* ($\mathbf{M}_h$ is dual to $\mathbf{W}_g$). Some examples of dual modulation and warping operators are listed in Table 7.2 (see also Table 7.1).

7.4 Affine Operators

The TF scaling operator $\mathbf{C}_\sigma$ and the time-shift operator $\mathbf{T}_\tau$ are unitary representations of the groups $(\mathbb{R}^+,\cdot)$ and $(\mathbb{R},+)$, respectively, on the Hardy space $H^2(\mathbb{R})$, and they satisfy the commutation relation

$$\mathbf{T}_{\sigma\tau}\mathbf{C}_\sigma=\mathbf{C}_\sigma\mathbf{T}_\tau\,. \tag{7.4.28}$$

(We note that the Hardy space $H^2(\mathbb{R})$ is a subspace of $L^2(\mathbb{R})$ that contains all functions whose Fourier transforms are zero for negative frequencies.) This relation between $\mathbf{T}_\tau$ and $\mathbf{C}_\sigma$ is generalized by the concept of *affine operators*. In this section, we introduce affine operators, discuss some of their properties, and show that suitably defined pairs of modulation and warping operators are affine operators.

Definition 7.8 *Let* $(\mathcal{G},\star)$ *and* $(\mathcal{H},\diamond)$ *be two groups isomorphic to* $(\mathbb{R},+)$ *with isomorphisms* $\psi_{\mathcal{G}}$ *and* $\psi_{\mathcal{H}}$, *respectively, and consider two unitary representations* $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ *and* $\{\mathbf{H}_h\}_{h\in\mathcal{H}}$ *of* $(\mathcal{G},\star)$ *and* $(\mathcal{H},\diamond)$, *respectively, on* *a minimal invariant Hilbert space $\mathcal{X}$. Then, $\{\mathbf{H}_h\}_{h\in\mathcal{H}}$ is called* affine to $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ *if*

$$\mathbf{H}_{\mu(g,h)}\mathbf{G}_g = \mathbf{G}_g\mathbf{H}_h\,, \quad \textit{for all } g \in (\mathcal{G},\star),\ h \in (\mathcal{H},\diamond)\,,$$
$$\textit{with } \mu(g,h) = \psi_{\mathcal{H}}^{-1}\big(\psi_{\mathcal{H}}(h)\exp\big(\psi_{\mathcal{G}}(g)\big)\big)\,. \qquad (7.4.29)$$

An elementary example of affine operators is given by the time-shift operator $\mathbf{T}_\tau$ and the scaling operator $\mathbf{C}_\sigma$ ($\mathbf{T}_\tau$ is affine to $\mathbf{C}_\sigma$). Further examples will be considered in Subsection 7.4.2.

7.4.1 Properties of Affine Operators

The next theorem states a relation of affine operators with the affine group (see Subsection 7.2.1), thus explaining the term "affine operators." Later, we will see that this result implies that we can use the composition $\mathbf{D}_{g,h} = \mathbf{H}_h\mathbf{G}_g$ or $\mathbf{D}'_{g,h} = \mathbf{G}_g\mathbf{H}_h$ of affine operators as "displacement operators."

Theorem 7.9 *Let $(\mathcal{G},\star)$ and $(\mathcal{H},\diamond)$ be isomorphic to $(\mathbb{R},+)$ with isomorphisms $\psi_{\mathcal{G}}$ and $\psi_{\mathcal{H}}$, respectively. Let $\{\mathbf{H}_h\}_{h\in\mathcal{H}}$ be affine to $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ on a minimal invariant Hilbert space $\mathcal{X}$. Then $\mathbf{D}_{g,h} \triangleq \mathbf{H}_h\mathbf{G}_g$ is an irreducible and faithful unitary representation of the group $(\mathcal{G}\times\mathcal{H},\circ)$ that is isomorphic to the affine group with isomorphism $\psi_{\mathcal{G}\times\mathcal{H}}(g,h) = \big(\exp\big(\psi_{\mathcal{G}}(g)\big),\psi_{\mathcal{H}}(h)\big)$, on $\mathcal{X}$, i.e.,*

$$\mathbf{D}_{g_0,h_0} = \mathbf{I}\,, \qquad \mathbf{D}_{g_2,h_2}\mathbf{D}_{g_1,h_1} = \mathbf{D}_{g_1\star g_2,\,\mu(g_2,h_1)\,\diamond\, h_2}\,, \qquad (7.4.30)$$

with $\mu(g,h) = \psi_{\mathcal{H}}^{-1}\big(\psi_{\mathcal{H}}(h)\exp\big(\psi_{\mathcal{G}}(g)\big)\big)$. Similarly, $\mathbf{D}'_{g,h} \triangleq \mathbf{G}_g\mathbf{H}_h$ is an irreducible and faithful unitary representation of the group $(\mathcal{G}\times\mathcal{H},\circ')$ that is isomorphic to the affine group with isomorphism $\psi_{\mathcal{G}\times\mathcal{H}}(g,h) = \big(\exp\big(\psi_{\mathcal{G}}(g)\big),\psi_{\mathcal{H}}(h)\exp\big(\psi_{\mathcal{G}}(g)\big)\big)$, on $\mathcal{X}$, i.e.,

$$\mathbf{D}'_{g_0,h_0} = \mathbf{I}, \qquad \mathbf{D}'_{g_2,h_2}\mathbf{D}'_{g_1,h_1} = \mathbf{D}'_{g_1\star g_2,\,h_1\diamond\mu(g_1^{-1},h_2)}.$$

Proof. Clearly, $\mathbf{D}_{g_0,h_0} = \mathbf{H}_{h_0}\mathbf{G}_{g_0} = \mathbf{I}$. Furthermore, due to (7.4.29) and (7.2.8) there is

$$\begin{aligned}\mathbf{D}_{g_2,h_2}\mathbf{D}_{g_1,h_1} &= \mathbf{H}_{h_2}\mathbf{G}_{g_2}\mathbf{H}_{h_1}\mathbf{G}_{g_1} = \mathbf{H}_{h_2}\mathbf{H}_{\mu(g_2,h_1)}\mathbf{G}_{g_2}\mathbf{G}_{g_1}\\ &= \mathbf{H}_{\mu(g_2,h_1)\diamond h_2}\mathbf{G}_{g_1\star g_2} = \mathbf{D}_{g_1\star g_2,\,\mu(g_2,h_1)\diamond h_2}\,,\end{aligned}$$

which is the second equation in (7.4.30). Hence, $\mathbf{D}_{g,h}$ is a unitary representation of $(\mathcal{G}\times\mathcal{H},\circ)$ with operation $(g_1,h_1)\circ(g_2,h_2) = \big(g_1\star g_2,\ \mu(g_2,h_1)\diamond h_2\big)$. To see that this group is isomorphic to the affine group, we apply the isomorphism $(\tilde{g},\tilde{h}) = \psi_{\mathcal{G}\times\mathcal{H}}(g,h) = \big(\exp\big(\psi_{\mathcal{G}}(g)\big),\psi_{\mathcal{H}}(h)\big)$ to this group operation and obtain, after some intermediate steps, $\psi_{\mathcal{G}\times\mathcal{H}}\big((g_1,h_1)\circ(g_2,h_2)\big) = \Big(\exp\big(\psi_{\mathcal{G}}(g_1)\big)\exp\big(\psi_{\mathcal{G}}(g_2)\big),\ \psi_{\mathcal{H}}(h_1)\exp\big(\psi_{\mathcal{G}}(g_2)\big) + \psi_{\mathcal{H}}(h_2)\Big)$, or equivalently, $(\tilde{g}_1, \tilde{h}_1)\tilde{\circ}(\tilde{g}_2, \tilde{h}_2) = (\tilde{g}_1\tilde{g}_2, \tilde{h}_1\tilde{g}_2 + \tilde{h}_2)$, which is the group operation of the affine group. Finally, since $\mathcal{X}$ is minimal invariant under $\mathbf{G}_g$ and $\mathbf{H}_h$, it is also minimal invariant under the composition $\mathbf{D}_{g,h} = \mathbf{H}_h\mathbf{G}_g$; hence, $\mathbf{D}_{g,h}$ is irreducible. The faithfulness of $\mathbf{D}_{g,h}$ follows from Theorem 7.10. The proof for $\mathbf{D}'_{g,h}$ is analogous. □

The next theorem states a relation of affine operators to the operators $\mathbf{C}_\sigma$ and $\mathbf{T}_\tau$.

Theorem 7.10 *Let $(\mathcal{G}, \star)$ and $(\mathcal{H}, \diamond)$ be isomorphic to $(\mathbb{R}, +)$ with isomorphisms $\psi_\mathcal{G}$ and $\psi_\mathcal{H}$, respectively, and let $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ and $\{\mathbf{H}_h\}_{h\in\mathcal{H}}$ be two unitary representations on a minimal invariant Hilbert space $\mathcal{X}$. Then, $\{\mathbf{H}_h\}_{h\in\mathcal{H}}$ is affine to $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ if and only if $\mathbf{G}_g$ and $\mathbf{H}_h$ are unitarily equivalent, up to parameter transformations specified below, to (respectively) $\mathbf{C}_\sigma$ and $\mathbf{T}_\tau$ on $H^2(\mathbb{R})$, i.e.,*

$$\mathbf{G}_g = \mathbf{V}\mathbf{C}_{\exp(\psi_\mathcal{G}(g))}\mathbf{V}^{-1} \qquad \textit{and} \qquad \mathbf{H}_h = \mathbf{V}\mathbf{T}_{\pm\psi_\mathcal{H}(h)}\mathbf{V}^{-1}\,, \tag{7.4.31}$$

where $\mathbf{V} : H^2(\mathbb{R}) \to \mathcal{X}$ is an invertible and isometric operator.

Proof. Using (7.4.28) and $\mathbf{V}^{-1}\mathbf{V} = \mathbf{I}$, Eq. (7.4.31) implies

$$\begin{aligned}\mathbf{G}_g\mathbf{H}_h &= \mathbf{V}\mathbf{C}_{\exp(\psi_\mathcal{G}(g))}\mathbf{V}^{-1}\mathbf{V}\mathbf{T}_{\pm\psi_\mathcal{H}(h)}\mathbf{V}^{-1} = \mathbf{V}\mathbf{C}_{\exp(\psi_\mathcal{G}(g))}\mathbf{T}_{\pm\psi_\mathcal{H}(h)}\mathbf{V}^{-1} \\ &= \mathbf{V}\mathbf{T}_{\pm\psi_\mathcal{H}(h)\exp(\psi_\mathcal{G}(g))}\mathbf{C}_{\exp(\psi_\mathcal{G}(g))}\mathbf{V}^{-1} = \mathbf{H}_{\mu(g,h)}\mathbf{G}_g\,,\end{aligned}$$

which is equal to (7.4.29). Hence, $\mathbf{H}_h$ is affine to $\mathbf{G}_g$.

Conversely, let $\{\mathbf{H}_h\}_{h\in\mathcal{H}}$ be affine to $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$. Due to Theorem 7.9, $\mathbf{D}_{g,h} = \mathbf{H}_h\mathbf{G}_g$ is an irreducible unitary representation of a group isomorphic to the affine group. According to [53, 54], there then exists an invertible and isometric operator $\mathbf{V} : H^2(\mathbb{R}) \to \mathcal{X}$ such that $\mathbf{D}_{g,h} = \mathbf{V}\mathbf{T}_{\pm\psi_\mathcal{H}(h)}\mathbf{C}_{\exp(\psi_\mathcal{G}(g))}\mathbf{V}^{-1}$ or, equivalently, $\mathbf{H}_h\mathbf{G}_g = \mathbf{V}\mathbf{T}_{\pm\psi_\mathcal{H}(h)}\mathbf{V}^{-1}\mathbf{V}\,\mathbf{C}_{\exp(\psi_\mathcal{G}(g))}\mathbf{V}^{-1}$. With $\mathbf{T}_{\pm\psi_\mathcal{H}(h_0)} = \mathbf{C}_{\exp(\psi_\mathcal{G}(g_0))} = \mathbf{I}$, this implies

$$\begin{aligned}\mathbf{G}_g &= \mathbf{H}_{h_0}\mathbf{G}_g = \mathbf{V}\mathbf{I}\mathbf{V}^{-1}\mathbf{V}\mathbf{C}_{\exp(\psi_\mathcal{G}(g))}\mathbf{V}^{-1} = \mathbf{V}\mathbf{C}_{\exp(\psi_\mathcal{G}(g))}\mathbf{V}^{-1} \\ \mathbf{H}_h &= \mathbf{H}_h\mathbf{G}_{g_0} = \mathbf{V}\mathbf{T}_{\pm\psi_\mathcal{H}(h)}\mathbf{V}^{-1}\mathbf{V}\mathbf{I}\mathbf{V}^{-1} = \mathbf{V}\mathbf{T}_{\pm\psi_\mathcal{H}(h)}\mathbf{V}^{-1}.\end{aligned}$$ □

We note that for $\mathcal{X} = L^2(\Omega)$, the operator $\mathbf{V}$ is given by $\mathbf{V} = \widetilde{\mathbf{U}}\mathbf{L}$, where $\widetilde{\mathbf{U}}$ was defined in (7.3.26) and $\mathbf{L}$ is given in the frequency domain (the frequency-domain version of $\mathbf{L}$ will be denoted as $\hat{\mathbf{L}}$) by $\mathbf{L} : H^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(\hat{\mathbf{L}}X)(f) = \sqrt{e^{-f}}\,X(e^{-f})$. This can be used to construct an "affine operator" $\mathbf{H}_h$ to a given operator $\mathbf{G}_g$: Let $\{\mathbf{G}_g\}_{g\in\mathcal{G}}$ be a unitary representation of $\mathcal{G}$ on $\mathcal{X}$ that is unitarily equivalent to $\mathbf{C}_\sigma$ as in (7.4.31). Then the eigenfunctions of $\mathbf{G}_g$ define the operator $\widetilde{\mathbf{U}}$ by (7.3.26), and $\mathbf{H}_h$ is obtained from the second equation in (7.4.31) with $\mathbf{V} = \widetilde{\mathbf{U}}\mathbf{L}$.

$m(t)$	Ω	$(\mathcal{H},\diamond)$ $(\mathcal{G},\star)$	$w_g(t)$	$(\mathbf{M}_h x)(t)$ $(\mathbf{W}_g x)(t)$
t	$\mathbb{R}$	$(\mathbb{R},+)$ $(\mathbb{R},+)$	$t\,e^g$	$e^{j2\pi ht}\,x(t)$ $e^{g/2}\,x(te^g)$
t	$\mathbb{R}^+$	$(\mathbb{R},+)$ $(\mathbb{R}^+,\cdot)$	t^g	$e^{j2\pi h \ln t}\,x(t)$ $\sqrt{g\,t^{g-1}}\,x(t^g)$
$\ln t$	$(1,\infty)$	$(\mathbb{R},+)$ $(\mathbb{R}^+,\cdot)$	$\exp\big((\ln t)^g\big)$	$e^{j2\pi h \ln(\ln t)}\,x(t)$ $\sqrt{w_g(t)\,\frac{g}{t}\,(\ln t)^{g-1}}\;x\big(w_g(t)\big)$
e^t	$\mathbb{R}$	$(\mathbb{R},+)$ $(\mathbb{R}^+,\cdot)$	$g\,t$	$e^{j2\pi ht}\,x(t)$ $\sqrt{g}\;x(gt)$
$\exp\big(\mathrm{sgn}(t)\lvert t\rvert^{1/\kappa}\big)$, $\kappa>0$	$\mathbb{R}$	$(\mathbb{R},+)$ $(\mathbb{R}^+,\cdot)$	$g^\kappa\,t$	$e^{j2\pi h\,\mathrm{sgn}(t)\lvert t\rvert^{1/\kappa}}\,x(t)$ $g^{\kappa/2}\,x(g^\kappa t)$

TABLE 7.3. *Some affine modulation and warping operators along with the underlying modulation and warping functions. We note that $m(\Omega)\subseteq\mathcal{G}$ such that $\tilde{m}(\Omega)=\mathbb{R}^+$, $\mathbb{R}^-$, or $\mathbb{R}$ (where $\tilde{m}(t)\triangleq\psi_{\mathcal{G}}(m(t))$), $g\in(\mathcal{G},\star)$, and $h\in(\mathcal{H},\diamond)$.*

7.4.2 Modulation and Warping Operators as Affine Operators

Let $(\mathcal{G},\star)$ and $(\mathcal{H},\diamond)$ be groups isomorphic to $(\mathbb{R},+)$ with isomorphisms $\psi_{\mathcal{G}}$ and $\psi_{\mathcal{H}}$, respectively. Consider the modulation operator $\{\mathbf{M}_h\}_{h\in(\mathcal{H},\diamond)}$ generated by a modulation function $m:\Omega\to\mathcal{G}$ with some set Ω, such that $\tilde{m}(\Omega)=\mathbb{R}^+$ or $\tilde{m}(\Omega)=\mathbb{R}^-$ or $\tilde{m}(\Omega)=\mathbb{R}$, where $\tilde{m}(t)\triangleq\psi_{\mathcal{G}}(m(t))$. Furthermore consider the warping operator $\{\mathbf{W}_g\}_{g\in(\mathcal{G},\star)}$ whose warping function is related to $m(t)$ according to

$$w_g(t) \;=\; \tilde{m}^{-1}\big(\tilde{m}(t)\exp\big(\psi_{\mathcal{G}}(g)\big)\big)\,. \tag{7.4.32}$$

Both $\mathbf{M}_h$ and $\mathbf{W}_g$ are defined on the Hilbert space $\mathcal{X}=L^2(\Omega)$. Note that $\mathbf{M}_h$ and $\mathbf{W}_g$ are *not* dual operators—indeed, the warping operator to which $\mathbf{M}_h$ is dual would have the warping function $w_g(t)=\tilde{m}^{-1}\big(\tilde{m}(t)-\psi_{\mathcal{G}}(g)\big)$. Instead, $\mathbf{M}_h$ is *affine* to $\mathbf{W}_g$, as is easily verified: we have

$$\begin{aligned}\big(\mathbf{M}_{\mu(g,h)}\mathbf{W}_g\,x\big)(t) \;&=\; e^{j2\pi\,\psi_{\mathcal{H}}(h)\exp(\psi_{\mathcal{G}}(g))\,\tilde{m}(t)}\,\sqrt{\lvert w_g'(t)\rvert}\;x\big(w_g(t)\big)\\ &=\; \sqrt{\lvert w_g'(t)\rvert}\;e^{j2\pi\,\psi_{\mathcal{H}}(h)\,\tilde{m}(w_g(t))}\,x\big(w_g(t)\big) \;=\; \big(\mathbf{W}_g\mathbf{M}_h x\big)(t)\,.\end{aligned}$$

Affine modulation and warping operators as formulated above are unitarily equivalent to $\mathbf{T}_\tau$ and $\mathbf{C}_\sigma$ (defined on $H^2(\mathbb{R})$) according to

$$\mathbf{M}_h=\mathbf{V}\mathbf{T}_{\pm\psi_{\mathcal{H}}(h)}\mathbf{V}^{-1} \quad\text{and}\quad \mathbf{W}_g=\mathbf{V}\mathbf{C}_{\exp(\psi_{\mathcal{G}}(g))}\mathbf{V}^{-1}\,, \tag{7.4.33}$$

where $\mathbf{V}=\mathbf{U}\mathbb{F}$ with $(\mathbf{U}x)(t)=\sqrt{\lvert\tilde{m}'(t)\rvert}\,x\big(\lvert\tilde{m}(t)\rvert\big)$. In (7.4.33), the $+$ $(-)$ sign applies for $\tilde{m}(\Omega)=\mathbb{R}^-$ $(\tilde{m}(\Omega)=\mathbb{R}^+)$. Some examples of affine modulation and warping operators are listed in Table 7.3 (cf. Table 7.1).

7.5 Covariant Signal Representations: Group Domain

So far, we have discussed modulation and warping operators as well as dual and affine pairs of operators. In this section, we begin our development of the covariance theory. This theory is based on unitary "displacement operators" (DOs) $\mathbf{D}_{\alpha,\beta}$, where the "displacement parameter" (α, β) belongs to a group. After a formal definition and discussion of DOs, we shall consider the systematic construction of linear and bilinear signal representations that are covariant to a given DO $\mathbf{D}_{\alpha,\beta}$. These covariant signal representations are functions of the displacement parameter (α, β). In later sections, we will consider the conversion of covariant (α, β)-representations into covariant TF representations.

7.5.1 Displacement Operators

The notion underlying DOs $\mathbf{D}_{\alpha,\beta}$ is that they displace signals in the TF plane without changing their energy. More specifically, if a signal x is concentrated about a TF point (t, f), then the transformed signal $\mathbf{D}_{\alpha,\beta}\, x$ is concentrated about a TF point (t', f') that depends on (t, f) and on (α, β). Elementary examples of DOs are the operators $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ and $\mathbf{R}_{\sigma,\tau} = \mathbf{T}_\tau \mathbf{C}_\sigma$ for which $(t', f') = (t + \tau, f + \nu)$ and $(t', f') = (\sigma t + \tau, f/\sigma)$, respectively. In what follows, we will often use the shorthand notation $\theta \triangleq (\alpha, \beta)$, so that DOs $\mathbf{D}_{\alpha,\beta}$ will be briefly written as $\mathbf{D}_\theta$.

We first discuss some basic assumptions about TF displacements and DOs as well as the mathematical structure of DOs that results from these assumptions.

1. We assume that, for each θ, the operator $\mathbf{D}_\theta$ maps a Hilbert space $\mathcal{X}$ onto itself. We also assume that $\mathbf{D}_\theta$ does not change a signal's energy and that all displacements can be reversed; hence, it is natural to model $\mathbf{D}_\theta$ as a *unitary* (i.e., invertible and isometric) operator on $\mathcal{X}$.

2. We further assume that displacing a signal first by θ_1 and then by θ_2 is essentially equivalent to a single displacement by $\theta_1 \circ \theta_2$, where $\circ$ is some group law; hence, $\theta = (\alpha, \beta)$ must belong to some group $(\mathcal{D}, \circ)$. In terms of $\mathbf{D}_\theta$, the above assumption suggests the composition law $\mathbf{D}_{\theta_2}\mathbf{D}_{\theta_1} = \mathbf{D}_{\theta_1 \circ \theta_2}$, i.e., that $\mathbf{D}_\theta$ is a unitary representation of the group $(\mathcal{D}, \circ)$. However, for the TF shift operator we have $\mathbf{S}_{\tau_2,\nu_2}\mathbf{S}_{\tau_1,\nu_1} = e^{-j2\pi\nu_1\tau_2}\, \mathbf{S}_{\tau_1+\tau_2,\nu_1+\nu_2}$. Allowing for a phase factor in our composition law implies that $\mathbf{D}_\theta$ is a a *projective* representation of the group $(\mathcal{D}, \circ)$ (see Subsection 7.2.2).

3. We also assume that the projective group representation $\mathbf{D}_\theta$ is *irreducible* on $\mathcal{X}$, i.e., $\mathcal{X}$ is minimal invariant under $\mathbf{D}_\theta$. Translated

into the TF domain, this corresponds to the requirement that all TF points (of our underlying TF point set) can be reached via displacements from other TF points.

4. Finally, we assume $\mathbf{D}_\theta$ to be a *faithful* group representation, i.e., $\mathbf{D}_\theta = \mathbf{D}_{\theta'}$ implies $\theta = \theta'$. This means that each TF displacement corresponds to a uniquely defined TF displacement parameter θ.

These assumptions will now be summarized in the following formal definition of a DO.

Definition 7.11 *[25, 28] Let $(\mathcal{D}, \circ)$ be a topological group with group identity element θ_0. We call a unitary operator family $\{\mathbf{D}_\theta\}_{\theta \in \mathcal{D}}$ defined on an infinite-dimensional Hilbert space $\mathcal{X}$ a* displacement operator (DO) *if $\mathbf{D}_\theta$ is an irreducible and faithful projective representation of $(\mathcal{D}, \circ)$ on $\mathcal{X}$. In particular, this implies that $\mathcal{X}$ is minimal invariant under $\mathbf{D}_\theta$ and that*

$$\mathbf{D}_{\theta_0} = \mathbf{I} \qquad \textit{and} \qquad \mathbf{D}_{\theta_2}\mathbf{D}_{\theta_1} = c(\theta_1, \theta_2)\, \mathbf{D}_{\theta_1 \circ \theta_2} \ \textit{for all } \theta_1, \theta_2 \in \mathcal{D}\,, \tag{7.5.34}$$

with a continuous function $c : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$.

An important example of DOs is provided by the composition of dual or affine warping and modulation operators based on groups $(\mathcal{A}, \bullet)$ and $(\mathcal{B}, *)$ that are isomorphic to $(\mathbb{R}, +)$:

$$(\mathbf{D}_{\alpha,\beta}\, x)(t) \;=\; (\mathbf{M}_\beta \mathbf{W}_\alpha\, x)(t) \;=\; e^{j2\pi\, \psi_\mathcal{B}(\beta)\, \tilde{m}(t)} \sqrt{|w'_\alpha(t)|}\; x\big(w_\alpha(t)\big)\,, \qquad \alpha \in (\mathcal{A}, \bullet), \;\; \beta \in (\mathcal{B}, *), \;\; t \in \Omega\,, \tag{7.5.35}$$

with $\tilde{m}(t) = \psi_\mathcal{A}(m(t))$. Here, $\mathbf{M}_\beta$ is either dual to $\mathbf{W}_\alpha$, i.e., $w_\alpha(t) = m^{-1}\big(m(t) \bullet \alpha^{-1}\big) = \tilde{m}^{-1}\big(\tilde{m}(t) - \psi_\mathcal{A}(\alpha)\big)$, or $\mathbf{M}_\beta$ is affine to $\mathbf{W}_\alpha$, i.e., $w_\alpha(t) = \tilde{m}^{-1}\big(\tilde{m}(t) \exp(\psi_\mathcal{A}(\alpha))\big)$. The cocycle of $\mathbf{D}_{\alpha,\beta}$ is $c\big((\alpha_1, \beta_1), (\alpha_2, \beta_2)\big) = \exp\big(-j2\pi\psi_\mathcal{A}(\alpha_2)\psi_\mathcal{B}(\beta_1)\big)$ in the dual case and $c\big((\alpha_1, \beta_1), (\alpha_2, \beta_2)\big) \equiv 1$ in the affine case.

The DO property is preserved by unitary equivalence transformations and invertible parameter transformations. Let $\{\mathbf{D}_\theta\}_{\theta \in (\mathcal{D}, \circ)}$ be a DO for a group $(\mathcal{D}, \circ)$ on a Hilbert space $\mathcal{X}$. Let $\mathbf{U}$ be an invertible and isometric operator mapping $\mathcal{X}$ onto a Hilbert space $\mathcal{X}'$ (generally different from $\mathcal{X}$), and let $(\mathcal{D}', \circ')$ be a group isomorphic to $(\mathcal{D}, \circ)$ via some isomorphism ψ. Then, the operator family

$$\mathbf{D}'_{\theta'} = \mathbf{U}\mathbf{D}_{\psi(\theta')}\mathbf{U}^{-1}\,, \qquad \theta' \in (\mathcal{D}', \circ') \tag{7.5.36}$$

can be shown to be a DO for the group $(\mathcal{D}', \circ')$, on the Hilbert space $\mathcal{X}'$.

Further important properties of DOs will be considered in Subsections 7.6.2 and 7.6.3.

7.5.2 Covariant Linear θ-Representations

Let us now consider linear θ-representations $L_x(\theta) = L_x(\alpha, \beta)$, i.e., signal representations that are functions of the displacement parameter $\theta \in (\mathcal{D}, \circ)$ and that depend on the signal $x(t)$ in a linear manner. We note that any linear θ-representation can be written in the form

$$L_x(\theta) \,=\, \langle x, h_\theta \rangle \,=\, \int_t x(t)\, h_\theta^*(t)\, dt\,, \tag{7.5.37}$$

where $h_\theta(t)$ is a function that depends on the displacement parameter θ but not on the signal x.

Definition 7.12 *[25] Let $\{\mathbf{D}_\theta\}_{\theta \in (\mathcal{D},\circ)}$ be a DO with cocycle c, defined on a Hilbert space $\mathcal{X}$. A linear θ-representation $L_x(\theta)$ is called* covariant to the DO $\mathbf{D}_\theta$ *if for all $x \in \mathcal{X}$*

$$L_{\mathbf{D}_{\theta_1} x}(\theta) \,=\, c(\theta \circ \theta_1^{-1}, \theta_1)\, L_x(\theta \circ \theta_1^{-1})\,, \quad \textit{for all } \theta, \theta_1 \in \mathcal{D}\,. \tag{7.5.38}$$

Hence, the representation of the displaced signal $(\mathbf{D}_{\theta_1} x)(t)$ equals (possibly up to a phase factor $c(\theta \circ \theta_1^{-1}, \theta_1)$) the displaced representation of the original signal $x(t)$, $L_x(\theta \circ \theta_1^{-1})$. We note that there is some arbitrariness regarding the exact definition of the phase factor in (7.5.38). Using $c(\theta \circ \theta_1^{-1}, \theta_1)$ as in (7.5.38) has the advantage that the expression for $L_x(\theta)$ to be obtained in (7.5.40) does not contain a phase factor.

For example, in the case of the TF shift operator $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$, whose cocycle is $c\big((\tau_1, \nu_1), (\tau_2, \nu_2)\big) = e^{-j2\pi\nu_1\tau_2}$, the covariance property (7.5.38) becomes

$$L_{\mathbf{S}_{\tau_1,\nu_1} x}(\tau, \nu) \,=\, e^{-j2\pi\tau_1(\nu-\nu_1)}\, L_x(\tau-\tau_1, \nu-\nu_1)\,. \tag{7.5.39}$$

Further examples of covariance properties are listed in Table 7.4.

The following fundamental theorem characterizes all covariant linear θ-representations. (We note that in [25], the theorems of this section were first formulated and proved directly for TF representations rather than θ-representations. In the following sections, we will see that this is in fact equivalent.)

Theorem 7.13 *[25] All linear θ-representations covariant to a DO $\{\mathbf{D}_\theta\}_{\theta \in (\mathcal{D},\circ)}$ on a Hilbert space $\mathcal{X}$ are given by the expression*

$$L_x(\theta) \,=\, \langle x, \mathbf{D}_\theta h \rangle \,=\, \int_t x(t)\, (\mathbf{D}_\theta h)^*(t)\, dt\,, \qquad \theta \in \mathcal{D}\,, \tag{7.5.40}$$

where h is an arbitrary function in $\mathcal{X}$.

Proof. First we show that the linear θ-representation in (7.5.40) satisfies the covariance property (7.5.38):

Short-Time Fourier Transform	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = x(t-\alpha)\,, \quad \alpha \in (\mathbb{R}, +)$
$\mathbf{B}_\beta$	$(\mathbf{B}_\beta x)(t) = e^{j2\pi\beta t}\, x(t)\,, \quad \beta \in (\mathbb{R}, +)$
cocycle	$c(\theta_1, \theta_2) = e^{-j2\pi\beta_1\alpha_2}$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = e^{-j2\pi\beta_1\alpha_2}\, \mathbf{D}_{\alpha_1+\alpha_2,\, \beta_1+\beta_2}$
covariance property	$L_{\mathbf{D}_{\alpha_1,\beta_1} x}(\alpha, \beta) = e^{-j2\pi\alpha_1(\beta-\beta_1)}\, L_x(\alpha-\alpha_1, \beta-\beta_1)$
covariant lin. θ-repres.	$L_x(\alpha, \beta) = \int_{-\infty}^{\infty} x(t)\, h^*(t-\alpha)\, e^{-j2\pi\beta t}\, dt$

Wavelet Transform	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+, \cdot)$
$\mathbf{B}_\beta$	$(\mathbf{B}_\beta x)(t) = x(t-\beta)\,, \quad \beta \in (\mathbb{R}, +)$
cocycle	$c(\theta_1, \theta_2) \equiv 1$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = \mathbf{D}_{\alpha_1\alpha_2,\, \beta_1\alpha_2+\beta_2}$
covariance property	$L_{\mathbf{D}_{\alpha_1,\beta_1} x}(\alpha, \beta) = L_x\big(\frac{\alpha}{\alpha_1}, \frac{\beta-\beta_1}{\alpha_1}\big)$
covariant lin. θ-repres.	$L_x(\alpha, \beta) = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} x(t)\, h^*\big(\frac{t-\beta}{\alpha}\big)\, dt$

Hyperbolic Wavelet Transform	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+, \cdot)$
$\mathbf{B}_\beta$	$(\hat{\mathbf{B}}_\beta X)(f) = e^{-j2\pi\beta \ln f}\, X(f)\,, \quad \beta \in (\mathbb{R}, +)\,, \ f > 0$
cocycle	$c(\theta_1, \theta_2) = e^{-j2\pi\beta_1 \ln \alpha_2}$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = e^{-j2\pi\beta_1 \ln \alpha_2}\, \mathbf{D}_{\alpha_1\alpha_2,\, \beta_1+\beta_2}$
covariance property	$L_{\mathbf{D}_{\alpha_1,\beta_1} x}(\alpha, \beta) = e^{-j2\pi(\beta-\beta_1)\ln \alpha_1}\, L_x\big(\frac{\alpha}{\alpha_1}, \beta-\beta_1\big)$
covariant lin. θ-repres.	$L_x(\alpha, \beta) = \sqrt{\alpha} \int_0^{\infty} X(f)\, H^*(\alpha f)\, e^{j2\pi\beta \ln f}\, df$

Power Wavelet Transform	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+, \cdot)$
$\mathbf{B}_\beta$	$(\hat{\mathbf{B}}_\beta X)(f) = e^{-j2\pi\beta\xi_\kappa(f)}\, X(f)\,, \quad \beta \in (\mathbb{R}, +)\,,$ with $\xi_\kappa(f) = \mathrm{sgn}(f)\, \lvert f \rvert^\kappa$
cocycle	$c(\theta_1, \theta_2) \equiv 1$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = \mathbf{D}_{\alpha_1\alpha_2,\, \beta_1\xi_\kappa(\alpha_2)+\beta_2}$
covariance property	$L_{\mathbf{D}_{\alpha_1,\beta_1} x}(\alpha, \beta) = L_x\big(\frac{\alpha}{\alpha_1}, \frac{\beta-\beta_1}{\xi_\kappa(\alpha_1)}\big)$
covariant lin. θ-repres. (extended to $X(f)$ on $f \in \mathbb{R}$)	$L_x(\alpha, \beta) = \sqrt{\alpha} \int_{-\infty}^{\infty} X(f)\, H^*(\alpha f)\, e^{j2\pi\beta\, \xi_\kappa(f)}\, df$

TABLE 7.4. *Some covariant linear θ-representations. Note that $\theta = (\alpha, \beta)$, $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta \mathbf{A}_\alpha$, $X(f)$ denotes the Fourier transform of $x(t)$, and $\hat{\mathbf{B}}_\beta$ denotes the frequency-domain version of $\mathbf{B}_\beta$.*

$$
\begin{aligned}
L_{\mathbf{D}_{\theta_1} x}(\theta) &= \langle \mathbf{D}_{\theta_1} x, \mathbf{D}_\theta h \rangle \;=\; \big\langle x, \mathbf{D}_{\theta_1}^{-1} \mathbf{D}_\theta h \big\rangle \;=\; \big\langle x, c^*(\theta_1^{-1}, \theta_1)\, \mathbf{D}_{\theta_1^{-1}} \mathbf{D}_\theta h \big\rangle \\
&= \big\langle x, c^*(\theta_1^{-1}, \theta_1)\, c(\theta, \theta_1^{-1})\, \mathbf{D}_{\theta \circ \theta_1^{-1}} h \big\rangle \\
&= c(\theta_1^{-1}, \theta_1)\, c^*(\theta, \theta_1^{-1})\, L_x(\theta \circ \theta_1^{-1}) \;=\; c(\theta \circ \theta_1^{-1}, \theta_1)\, L_x(\theta \circ \theta_1^{-1})\,,
\end{aligned}
$$

where we have used the unitarity of $\mathbf{D}_\theta$ as well as (7.2.13), (7.2.12), (7.5.34), (7.2.10), and (7.2.11).

Conversely, let $L_x(\theta)$ satisfy the covariance property (7.5.38). With (7.5.37), Eq. (7.5.38) becomes $\langle \mathbf{D}_{\theta_1} x, h_\theta \rangle = c(\theta \circ \theta_1^{-1}, \theta_1)\, \big\langle x, h_{\theta \circ \theta_1^{-1}} \big\rangle$. Substituting $\theta = \theta_0$ and $\theta_1^{-1} = \theta$ yields $\langle \mathbf{D}_{\theta^{-1}} x, h_{\theta_0} \rangle = c(\theta_0 \circ \theta, \theta^{-1}) \langle x, h_{\theta_0 \circ \theta} \rangle = c(\theta, \theta^{-1}) \langle x, h_\theta \rangle$ and hence

$$\begin{aligned} L_x(\theta) &= \langle x, h_\theta \rangle = c^*(\theta, \theta^{-1}) \left\langle \mathbf{D}_{\theta^{-1}} x, h_{\theta_0} \right\rangle = |c(\theta, \theta^{-1})|^2 \left\langle \mathbf{D}_\theta^{-1} x, h_{\theta_0} \right\rangle \\ &= \left\langle \mathbf{D}_\theta^{-1} x, h_{\theta_0} \right\rangle = \left\langle x, \mathbf{D}_\theta h \right\rangle , \end{aligned}$$

with $h(t) \triangleq h_{\theta_0}(t)$. □

For example, the expression (7.5.40) of all linear θ-representations covariant to $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ becomes

$$L_x(\tau, \nu) = \langle x, \mathbf{S}_{\tau,\nu} h \rangle = \int_t x(t)\, h^*(t-\tau)\, e^{-j2\pi\nu t}\, dt , \qquad (7.5.41)$$

which is the short-time Fourier transform in (7.1.4). Further examples are listed in Table 7.4.

Let us now consider unitarily equivalent DOs $\mathbf{D}_\theta$ and $\mathbf{D}'_{\theta'} = \mathbf{U}\mathbf{D}_{\psi(\theta')}\mathbf{U}^{-1}$ on $\mathcal{X}$ and $\mathcal{X}'$, respectively (see (7.5.36)). By Theorem 7.13, the class of all linear θ'-representations covariant to $\mathbf{D}'_{\theta'}$ is given by

$$\begin{aligned} L'_x(\theta') &= \langle x, \mathbf{D}'_{\theta'} h' \rangle = \langle x, \mathbf{U}\mathbf{D}_{\psi(\theta')}\mathbf{U}^{-1} h' \rangle = \langle \mathbf{U}^{-1} x, \mathbf{D}_{\psi(\theta')}\mathbf{U}^{-1} h' \rangle \\ &= \langle \mathbf{U}^{-1} x, \mathbf{D}_{\psi(\theta')} h \rangle = L_{\mathbf{U}^{-1}x}(\psi(\theta')) , \end{aligned}$$

where $h = \mathbf{U}^{-1} h'$ is an arbitrary function in $\mathcal{X}$ and L is a linear θ-representation covariant to $\mathbf{D}_\theta$. Hence, there exists a simple relation between the covariant classes $\{L_x(\theta)\}$ and $\{L'_x(\theta')\}$.

7.5.3 Covariant Bilinear θ-Representations

Next, we consider bilinear θ-representations $B_{x,y}(\theta) = B_{x,y}(\alpha, \beta)$ with $\theta \in (\mathcal{D}, \circ)$. Any bilinear θ-representation can be written in the form [13]

$$B_{x,y}(\theta) = \langle x, \mathbf{K}_\theta\, y \rangle = \int_{t_1} \int_{t_2} x(t_1)\, y^*(t_2)\, k_\theta^*(t_1, t_2)\, dt_1 dt_2 , \qquad (7.5.42)$$

where $\mathbf{K}_\theta$ is a linear operator that depends on the displacement parameter θ but not on x, y, and $k_\theta(t_1, t_2)$ is its kernel. The following definition and theorem are analogous to the linear case previously discussed.

Definition 7.14 *[25, 26, 28, 31, 32] Let $\{\mathbf{D}_\theta\}_{\theta \in (\mathcal{D}, \circ)}$ be a DO defined on a Hilbert space $\mathcal{X}$. A bilinear θ-representation $B_{x,y}(\theta)$ is called* covariant to the DO $\mathbf{D}_\theta$ *if for all $x, y \in \mathcal{X}$*

$$B_{\mathbf{D}_{\theta_1} x, \mathbf{D}_{\theta_1} y}(\theta) = B_{x,y}(\theta \circ \theta_1^{-1}) , \quad \textit{for all } \theta, \theta_1 \in \mathcal{D} . \qquad (7.5.43)$$

We note that this definition of covariance differs from the covariance definition in the linear case, (7.5.38), by the absence of the DO's cocycle c. This difference is caused by the bilinear structure of B.

Theorem 7.15 *[25, 26, 28, 31, 32] All bilinear θ-representations covariant to a DO $\{\mathbf{D}_\theta\}_{\theta\in(\mathcal{D},\circ)}$ on a Hilbert space $\mathcal{X}$ are given by the expression*

$$\begin{aligned} B_{x,y}(\theta) &= \langle x, \mathbf{D}_\theta \mathbf{K} \mathbf{D}_\theta^{-1} y\rangle \\ &= \int_{t_1}\int_{t_2} x(t_1)\, y^*(t_2) \left[\mathbf{D}_\theta \mathbf{K} \mathbf{D}_\theta^{-1}\right]^*(t_1,t_2)\, dt_1 dt_2\,, \quad \theta\in\mathcal{D}, \quad (7.5.44) \end{aligned}$$

where $\mathbf{K}$ is an arbitrary linear operator on $\mathcal{X}$ and $\left[\mathbf{D}_\theta \mathbf{K} \mathbf{D}_\theta^{-1}\right](t_1,t_2)$ denotes the kernel of $\mathbf{D}_\theta \mathbf{K} \mathbf{D}_\theta^{-1}$.

Proof. We shall use the proof given in [31, 32] since it is simpler than the approach taken in [25]. First we show that the bilinear θ-representation in (7.5.44) satisfies the covariance property (7.5.43):

$$\begin{aligned} B_{\mathbf{D}_{\theta_1}x,\mathbf{D}_{\theta_1}y}(\theta) &= \left\langle \mathbf{D}_{\theta_1}x, \mathbf{D}_\theta \mathbf{K} \mathbf{D}_\theta^{-1} \mathbf{D}_{\theta_1} y\right\rangle = \left\langle x, \mathbf{D}_{\theta_1}^{-1}\mathbf{D}_\theta \mathbf{K} (\mathbf{D}_{\theta_1}^{-1}\mathbf{D}_\theta)^{-1} y\right\rangle \\ &= \left\langle x, c^*(\theta_1,\theta_1^{-1})\, \mathbf{D}_{\theta_1^{-1}} \mathbf{D}_\theta \mathbf{K} \left[c^*(\theta_1,\theta_1^{-1})\, \mathbf{D}_{\theta_1^{-1}} \mathbf{D}_\theta\right]^{-1} y\right\rangle \\ &= \left\langle x, \mathbf{D}_{\theta_1^{-1}} \mathbf{D}_\theta \mathbf{K} (\mathbf{D}_{\theta_1^{-1}} \mathbf{D}_\theta)^{-1} y\right\rangle \\ &= \left\langle x, c(\theta,\theta_1^{-1})\, \mathbf{D}_{\theta\circ\theta_1^{-1}} \mathbf{K} \left[c(\theta,\theta_1^{-1})\, \mathbf{D}_{\theta\circ\theta_1^{-1}}\right]^{-1} y\right\rangle = B_{x,y}(\theta\circ\theta_1^{-1})\,, \end{aligned}$$

where we have used the unitarity of $\mathbf{D}_\theta$ as well as (7.2.13) and (7.5.34).

Conversely, let $B_{x,y}(\theta)$ satisfy the covariance property (7.5.43). With (7.5.42), Eq. (7.5.43) becomes $\langle \mathbf{D}_{\theta_1}x, \mathbf{K}_\theta \mathbf{D}_{\theta_1} y\rangle = \langle x, \mathbf{K}_{\theta\circ\theta_1^{-1}} y\rangle$. The substitution $\theta=\theta_0$, $\theta_1^{-1}=\theta$ yields $\langle \mathbf{D}_{\theta^{-1}}x, \mathbf{K}_{\theta_0} \mathbf{D}_{\theta^{-1}} y\rangle = \langle x, \mathbf{K}_{\theta_0\circ\theta} y\rangle = \langle x, \mathbf{K}_\theta y\rangle$ and hence

$$\begin{aligned} B_{x,y}(\theta) &= \langle x, \mathbf{K}_\theta y\rangle = \langle \mathbf{D}_{\theta^{-1}}x, \mathbf{K}_{\theta_0} \mathbf{D}_{\theta^{-1}} y\rangle \\ &= |c(\theta,\theta^{-1})|^2\, \langle \mathbf{D}_\theta^{-1}x, \mathbf{K}_{\theta_0} \mathbf{D}_\theta^{-1} y\rangle = \langle x, \mathbf{D}_\theta \mathbf{K} \mathbf{D}_\theta^{-1} y\rangle, \end{aligned}$$

with $\mathbf{K} \triangleq \mathbf{K}_{\theta_0}$. □

For example, in the case of the TF shift operator $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$, the covariance property (7.5.43) reads

$$B_{\mathbf{S}_{\tau_1,\nu_1}x,\mathbf{S}_{\tau_1,\nu_1}y}(\tau,\nu) = B_{x,y}(\tau-\tau_1,\nu-\nu_1)\,,$$

and the canonical expression (7.5.44) of all bilinear θ-representations covariant to $\mathbf{S}_{\tau,\nu}$ becomes

$$\begin{aligned} B_{x,y}(\tau,\nu) &= \langle x, \mathbf{S}_{\tau,\nu}\mathbf{K}\mathbf{S}_{\tau,\nu}^{-1} y\rangle \\ &= \int_{t_1}\int_{t_2} x(t_1)\, y^*(t_2)\; k^*(t_1-\tau, t_2-\tau)\; e^{-j2\pi\nu(t_1-t_2)}\, dt_1 dt_2\,, \end{aligned}$$

which is recognized as Cohen's class in (7.1.6). Further examples of covariance properties and the resulting canonical expressions of covariant BTFRs are listed in Table 7.5.

Cohen's Class	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = x(t-\alpha)\,, \quad \alpha \in (\mathbb{R},+)$
$\mathbf{B}_\beta$	$(\mathbf{B}_\beta x)(t) = e^{j2\pi\beta t}\, x(t)\,, \quad \beta \in (\mathbb{R},+)$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = e^{-j2\pi\beta_1\alpha_2}\,\mathbf{D}_{\alpha_1+\alpha_2,\,\beta_1+\beta_2}$
covariance property	$B_{\mathbf{D}_{\alpha_1,\beta_1}x,\mathbf{D}_{\alpha_1,\beta_1}y}(\alpha,\beta) = B_{x,y}(\alpha-\alpha_1,\beta-\beta_1)$
covar. bilin. θ-repres.	$B_{x,y}(\alpha,\beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, y^*(t_2)\, k^*(t_1-\alpha,t_2-\alpha)$ $\cdot\, e^{-j2\pi\beta(t_1-t_2)}\, dt_1 dt_2$

Affine Class	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+,\cdot)$
$\mathbf{B}_\beta$	$(\mathbf{B}_\beta x)(t) = x(t-\beta)\,, \quad \beta \in (\mathbb{R},+)$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = \mathbf{D}_{\alpha_1\alpha_2,\,\beta_1\alpha_2+\beta_2}$
covariance property	$B_{\mathbf{D}_{\alpha_1,\beta_1}x,\mathbf{D}_{\alpha_1,\beta_1}y}(\alpha,\beta) = B_{x,y}\big(\frac{\alpha}{\alpha_1},\frac{\beta-\beta_1}{\alpha_1}\big)$
covar. bilin. θ-repres.	$B_{x,y}(\alpha,\beta) = \frac{1}{\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, y^*(t_2)$ $\cdot\, k^*\big(\frac{t_1-\beta}{\alpha},\frac{t_2-\beta}{\alpha}\big)\, dt_1 dt_2$

Hyperbolic Class	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+,\cdot)$
$\mathbf{B}_\beta$	$(\hat{\mathbf{B}}_\beta X)(f) = e^{-j2\pi\beta\ln f}\, X(f)\,, \quad \beta \in (\mathbb{R},+)\,,\ f>0$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = e^{-j2\pi\beta_1\ln\alpha_2}\,\mathbf{D}_{\alpha_1\alpha_2,\,\beta_1+\beta_2}$
covariance property	$B_{\mathbf{D}_{\alpha_1,\beta_1}x,\mathbf{D}_{\alpha_1,\beta_1}y}(\alpha,\beta) = B_{x,y}\big(\frac{\alpha}{\alpha_1},\beta-\beta_1\big)$
covar. bilin. θ-repres.	$B_{x,y}(\alpha,\beta) = \alpha\int_0^{\infty}\int_0^{\infty} X(f_1)\, Y^*(f_2)\, \hat{k}^*(\alpha f_1,\alpha f_2)$ $\cdot\, e^{j2\pi\beta\ln(f_1/f_2)}\, df_1 df_2$

Power Classes	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+,\cdot)$
$\mathbf{B}_\beta$	$(\hat{\mathbf{B}}_\beta X)(f) = e^{-j2\pi\beta\xi_\kappa(f)}\, X(f)\,, \quad \beta \in (\mathbb{R},+),$ with $\xi_\kappa(f) = \mathrm{sgn}(f)\,\lvert f\rvert^\kappa$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = \mathbf{D}_{\alpha_1\alpha_2,\,\beta_1\xi_\kappa(\alpha_2)+\beta_2}$
covariance property	$B_{\mathbf{D}_{\alpha_1,\beta_1}x,\mathbf{D}_{\alpha_1,\beta_1}y}(\alpha,\beta) = B_{x,y}\big(\frac{\alpha}{\alpha_1},\frac{\beta-\beta_1}{\xi_\kappa(\alpha_1)}\big)$
covar. bilin. θ-rep. (extended to $X(f), Y(f)$ on $f\in\mathbb{R}$)	$B_{x,y}(\alpha,\beta) = \alpha\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f_1)\, Y^*(f_2)\, \hat{k}^*(\alpha f_1,\alpha f_2)$ $\cdot\, e^{j2\pi\beta\,[\xi_\kappa(f_1)-\xi_\kappa(f_2)]}\, df_1 df_2$

TABLE 7.5. *Some covariant bilinear/quadratic θ-representations. Note that $\theta = (\alpha,\beta)$, $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$, $X(f)$ denotes the Fourier transform of $x(t)$, and $\hat{\mathbf{B}}_\beta$ denotes the frequency-domain version of $\mathbf{B}_\beta$.*

Using Theorem 7.15, one easily shows [12] that the bilinear θ-representations covariant to a DO $\mathbf{D}_\theta$ are related to the bilinear θ'-representations covariant to the DO $\mathbf{D}'_{\theta'} = \mathbf{U}\mathbf{D}_{\psi(\theta')}\mathbf{U}^{-1}$ (see (7.5.36)) according to

$$B'_{x,y}(\theta') \;=\; B_{\mathbf{U}^{-1}x,\mathbf{U}^{-1}y}(\psi(\theta'))\,,$$

where the operators $\mathbf{K}$ and $\mathbf{K}'$ used in $B_{x,y}(\theta) = \langle x, \mathbf{D}_\theta\mathbf{K}\mathbf{D}_\theta^{-1}y\rangle$ and in $B'_{x,y}(\theta') = \langle x, \mathbf{D}'_{\theta'}\mathbf{K}'\mathbf{D}'^{-1}_{\theta'}y\rangle$, respectively are related as $\mathbf{K} = \mathbf{U}^{-1}\mathbf{K}'\mathbf{U}$.

7.6 The Displacement Function: Theory

Thus far, we have characterized all linear and bilinear θ-representations that are covariant to a given DO $\{\mathbf{D}_\theta\}_{\theta\in(\mathcal{D},\circ)}$. However, ultimately we are interested in covariant *TF representations* that are functions of (t,f) rather than of $\theta=(\alpha,\beta)$. As we shall see, covariant θ-representations can be converted into covariant TF representations via a one-to-one mapping $\theta \leftrightarrow (t,f)$ that will be termed the *displacement function* (DF) [25, 26, 28]. In this section, we shall formulate basic axioms of TF displacements and derive some remarkable consequences regarding the mathematical structure of the DF as well as the underlying group and DO. A systematic method for constructing the DF and the use of the DF for constructing covariant TF representations will be described in Sections 7.7 and 7.8, respectively.

Hereafter, similarly to our shorthand notation $\theta=(\alpha,\beta)$, we shall frequently use the shorthand notation $z \triangleq (t,f)$ for TF points. Furthermore, $\mathcal{Z}\subseteq\mathbb{R}^2$ will denote the set of TF points $z=(t,f)$ on which a covariant TF representation $L_x(z)=L_x(t,f)$ or $B_{x,y}(z)=B_{x,y}(t,f)$ is defined.

7.6.1 Axioms of Time-Frequency Displacement

Let us consider the TF displacements performed by a given DO $\{\mathbf{D}_\theta\}_{\theta\in(\mathcal{D},\circ)}$. If a signal $x(t)$ is TF localized about some TF point $z_1=(t_1,f_1)\in\mathcal{Z}$, then the transformed ("displaced") signal $(\mathbf{D}_\theta\, x)(t)$ will be localized about some other TF point $z_2=(t_2,f_2)\in\mathcal{Z}$ that depends on z_1 and θ, i.e.,

$$z_2 = e(z_1,\theta)\,.$$

The function $e(\cdot,\cdot)$ will be called the *extended DF* of the DO $\mathbf{D}_\theta$ [28] (originally termed DF in [25, 26]). Before discussing the construction of the extended DF in Section 7.7, we formulate three axioms that the extended DF is assumed to satisfy and study important consequences of these axioms.

Definition 7.16 *Let $(\mathcal{D},\circ)$ with $\mathcal{D}\subseteq\mathbb{R}^2$ be a group and $\mathcal{Z}\subseteq\mathbb{R}^2$ an open, simply connected set of TF points $z=(t,f)$. ("Simply connected" means that $\mathcal{Z}$ has no "holes" [53].) A mapping $e:\mathcal{Z}\times\mathcal{D}\to\mathcal{Z}$ is called an* extended DF *for the group $\mathcal{D}$ on the TF set $\mathcal{Z}$ if it satisfies the following axioms:*

1. *For any fixed $\theta\in\mathcal{D}$, $z\mapsto z'=e(z,\theta)$ is an invertible, continuously differentiable, and area-preserving mapping of $\mathcal{Z}$ onto $\mathcal{Z}$.*
2. *For any fixed $z\in\mathcal{Z}$, $\theta\mapsto z'=e(z,\theta)$ is an invertible, continuously differentiable mapping of $\mathcal{D}$ onto $\mathcal{Z}$ with nonzero Jacobian, i.e., a* diffeomorphism *[53]. (We note that this property presupposes that $(\mathcal{D},\circ)$ is a* topological *group with the Euclidean topology.)*
3. *Composition property:*

$$e\big(e(z,\theta_1),\theta_2\big) = e(z,\theta_1\circ\theta_2) \quad \text{for all } \theta_1,\theta_2\in\mathcal{D}\,. \tag{7.6.45}$$

Furthermore, if these axioms are satisfied, the function

$$d(\theta) \triangleq e(z_0, \theta),$$

with $z_0 = (t_0, f_0) \in \mathcal{Z}$ *an arbitrary but fixed reference TF point, will be called a* displacement function (DF) *associated to the extended DF* e.

These axioms imply that $(\mathcal{D}, \circ)$ acts as a *transitive transformation group* [55] on $\mathcal{Z}$. According to the first two axioms, to any $\theta \in \mathcal{D}$ and $z' \in \mathcal{Z}$ there exists a unique $z \in \mathcal{Z}$ such that $z' = e(z, \theta)$; similarly, to any $z, z' \in \mathcal{Z}$ there exists a unique $\theta \in \mathcal{D}$ such that $z' = e(z, \theta)$. Axiom 3 means that displacing a signal first by θ_1 and then by θ_2 is equivalent to a single displacement by $\theta_1 \circ \theta_2$; this is the counterpart of the DO composition property (7.5.34), $\mathbf{D}_{\theta_2}\mathbf{D}_{\theta_1} = c(\theta_1, \theta_2)\,\mathbf{D}_{\theta_1 \circ \theta_2}$. It can also be shown that $e(z, \theta_0) = z$, i.e., the group identity element θ_0 corresponds to no displacement.

Theorem 7.17 *Consider an extended DF* $e(z, \theta)$ *for a group* $(\mathcal{D}, \circ)$ *on a TF set* $\mathcal{Z}$. *Any associated DF* $d(\theta)$ *is a diffeomorphism, i.e., an invertible, continuously differentiable mapping of* $\mathcal{D}$ *onto* $\mathcal{Z}$ *with nonzero Jacobian. Moreover,* e *can be written in terms of* d *and its inverse* d^{-1} *as*

$$e(z, \theta) = d\big(d^{-1}(z) \circ \theta\big). \tag{7.6.46}$$

Proof. Let $d(\theta) = e(z_0, \theta)$ with $z_0 \in \mathcal{Z}$ fixed. Axiom 2 in Definition 7.16 implies that d is a diffeomorphism from $\mathcal{D}$ onto $\mathcal{Z}$. It follows that the inverse function d^{-1} exists, so we have $z = d\big(d^{-1}(z)\big) = e\big(z_0, d^{-1}(z)\big)$ and further

$$e(z, \theta) = e\big(e\big(z_0, d^{-1}(z)\big), \theta\big) = e\big(z_0, d^{-1}(z) \circ \theta\big) = d\big(d^{-1}(z) \circ \theta\big),$$

where (7.6.45) has been used. □

Finally, consider two different DFs $d(\theta) = e(z_0, \theta)$ and $\tilde{d}(\theta) = e(\tilde{z}_0, \theta)$ associated to the same extended DF e. Due to Axiom 2 in Definition 7.16, there is a unique $\tilde{\theta} \in \mathcal{D}$ such that $\tilde{z}_0 = e(z_0, \tilde{\theta})$. This yields

$$\tilde{d}(\theta) = e(\tilde{z}_0, \theta) = e\big(e(z_0, \tilde{\theta}), \theta\big) = e(z_0, \tilde{\theta} \circ \theta) = d(\tilde{\theta} \circ \theta),$$

or, equivalently formulated with the inverse DFs,

$$\tilde{d}^{-1}(z) = \tilde{\theta}^{-1} \circ d^{-1}(z). \tag{7.6.47}$$

7.6.2 Lie Group Structure and Consequences

The existence of a DF has important consequences regarding the structure of the underlying group $(\mathcal{D}, \circ)$ and DO $\{\mathbf{D}_\theta\}_{\theta \in (\mathcal{D}, \circ)}$. In fact, as shown in the proof of the following theorem, the group must be a *simply connected Lie group*, i.e., a topological group that can be viewed as a smooth surface in two-dimensional (2-D) Euclidean space. Since there exist only two simply connected 2-D Lie groups (up to isomorphisms), namely, the group $(\mathbb{R}^2, +)$ and the affine group [53], we obtain the following result.

Theorem 7.18 *Let $(\mathcal{D}, \circ)$ be a group with $\mathcal{D} \subseteq \mathbb{R}^2$ and let the set $\mathcal{Z} \subseteq \mathbb{R}^2$ be open and simply connected. If there exists an extended DF $e: \mathcal{Z} \times \mathcal{D} \to \mathcal{Z}$, then either $(\mathcal{D}, \circ)$ is isomorphic to $(\mathbb{R}^2, +)$, i.e., the group $(\mathcal{D}', \circ')$ with $\mathcal{D}' = \mathbb{R}^2$ and $(\alpha_1', \beta_1') \circ' (\alpha_2', \beta_2') = (\alpha_1' + \alpha_2', \beta_1' + \beta_2')$, or $(\mathcal{D}, \circ)$ is isomorphic to the affine group, i.e., the group $(\mathcal{D}', \circ')$ with $\mathcal{D}' = \mathbb{R}^+ \times \mathbb{R}$ and $(\alpha_1', \beta_1') \circ' (\alpha_2', \beta_2') = (\alpha_1' \alpha_2', \, \beta_1' \alpha_2' + \beta_2')$.*

Proof. According to Axiom 2 in Definition 7.16, the existence of an extended DF presupposes that $(\mathcal{D}, \circ)$ is a topological group with the Euclidean topology on $\mathcal{D} \subseteq \mathbb{R}^2$. According to [53, 55], this implies that $(\mathcal{D}, \circ)$ is a Lie group. Since $\mathcal{Z}$ is simply connected and the DF $d: \mathcal{D} \to \mathcal{Z}$ is a homeomorphism (see Theorem 7.17, noting that a diffeomorphism is always a homeomorphism), also $\mathcal{D}$ is simply connected. Hence, $(\mathcal{D}, \circ)$ is a simply connected 2-D Lie group. It is well known [53] that there exist only two simply connected 2-D Lie groups (up to isomorphisms), namely, the group $(\mathbb{R}^2, +)$ and the affine group. This proves the theorem. □

This result has an important consequence regarding the DO $\{\mathbf{D}_\theta\}_{\theta \in (\mathcal{D}, \circ)}$. The next two corollaries state that for the two possible cases—i.e., $(\mathcal{D}, \circ)$ isomorphic to $(\mathbb{R}^2, +)$ or the affine group—the DO has a *separable* structure. (Note the consistency of these corollaries with Theorems 7.6 and 7.9.)

Corollary 7.19 *Consider a group $(\mathcal{D}, \circ)$ that is isomorphic to $(\mathbb{R}^2, +)$ with isomorphism $\psi: \mathcal{D} \to \mathbb{R}^2$. If $\{\mathbf{D}'_{\theta'}\}_{\theta' \in (\mathcal{D}, \circ)}$ is a DO for $(\mathcal{D}, \circ)$, with cocycle $c'\big((\alpha_1', \beta_1'), (\alpha_2', \beta_2')\big)$, then $\mathbf{D}_\theta \triangleq \mathbf{D}'_{\psi^{-1}(\theta)}$ is a DO for $(\mathbb{R}^2, +)$, with cocycle $c\big((\alpha_1, \beta_1), (\alpha_2, \beta_2)\big) = c'\big(\psi^{-1}(\alpha_1, \beta_1), \psi^{-1}(\alpha_2, \beta_2)\big)$, and it is* separable *(up to a phase factor) according to*

$$\mathbf{D}_\theta \equiv \mathbf{D}_{\alpha,\beta} = c^*\big((\alpha, 0), (0, \beta)\big)\, \mathbf{B}_\beta \mathbf{A}_\alpha \quad \text{with } \mathbf{A}_\alpha \triangleq \mathbf{D}_{\alpha,0}, \;\; \mathbf{B}_\beta \triangleq \mathbf{D}_{0,\beta}. \tag{7.6.48}$$

Here, $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are faithful projective representations of $(\mathbb{R}, +)$; they satisfy

$$\mathbf{A}_0 = \mathbf{I}, \qquad \mathbf{A}_{\alpha_2} \mathbf{A}_{\alpha_1} = c_{\mathbf{A}}(\alpha_1, \alpha_2)\, \mathbf{A}_{\alpha_1 + \alpha_2} \quad \text{for all } \alpha_1, \alpha_2 \in \mathbb{R}$$
$$\mathbf{B}_0 = \mathbf{I}, \qquad \mathbf{B}_{\beta_2} \mathbf{B}_{\beta_1} = c_{\mathbf{B}}(\beta_1, \beta_2)\, \mathbf{B}_{\beta_1 + \beta_2} \quad \text{for all } \beta_1, \beta_2 \in \mathbb{R},$$

with $c_{\mathbf{A}}(\alpha_1, \alpha_2) = c\big((\alpha_1, 0), (\alpha_2, 0)\big)$ and $c_{\mathbf{B}}(\beta_1, \beta_2) = c\big((0, \beta_1), (0, \beta_2)\big)$. If $c\big((\alpha_1, \beta_1), (\alpha_2, \beta_2)\big) = e^{-j2\pi\alpha_2\beta_1}$, then $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta \mathbf{A}_\alpha$, and $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are unitary representations of $(\mathbb{R}, +)$ with $\mathbf{B}_\beta$ dual to $\mathbf{A}_\alpha$.

Proof. If $\mathbf{D}'_{\theta'}$ is a DO for the group $(\mathcal{D}, \circ)$, then $\mathbf{D}_\theta = \mathbf{D}'_{\psi^{-1}(\theta)}$ satisfies $\mathbf{D}_{0,0} = \mathbf{D}'_{\psi^{-1}(0,0)} = \mathbf{D}'_{\theta_0'} = \mathbf{I}$ and

$$\begin{aligned}
\mathbf{D}_{\alpha_2,\beta_2} \mathbf{D}_{\alpha_1,\beta_1} &= \mathbf{D}'_{\psi^{-1}(\alpha_2,\beta_2)} \mathbf{D}'_{\psi^{-1}(\alpha_1,\beta_1)} \\
&= c'\big(\psi^{-1}(\alpha_1, \beta_1), \psi^{-1}(\alpha_2, \beta_2)\big)\, \mathbf{D}'_{\psi^{-1}(\alpha_1,\beta_1) \circ \psi^{-1}(\alpha_2,\beta_2)} \\
&= c'\big(\psi^{-1}(\alpha_1, \beta_1), \psi^{-1}(\alpha_2, \beta_2)\big)\, \mathbf{D}'_{\psi^{-1}(\alpha_1+\alpha_2, \, \beta_1+\beta_2)}
\end{aligned}$$

$$= c\big((\alpha_1,\beta_1),(\alpha_2,\beta_2)\big)\,\mathbf{D}_{\alpha_1+\alpha_2,\,\beta_1+\beta_2}\,,$$

where $c\big((\alpha_1,\beta_1),(\alpha_2,\beta_2)\big) = c'\big(\psi^{-1}(\alpha_1,\beta_1),\psi^{-1}(\alpha_2,\beta_2)\big)$. The cocycle properties are easily shown to be satisfied. Hence, $\mathbf{D}_\theta$ is a projective representation of $(\mathbb{R}^2,+)$. It is irreducible and faithful because $\mathbf{D}'_{\psi^{-1}(\theta)}$ is; therefore, it is a DO for $(\mathbb{R}^2,+)$. The separability of $\mathbf{D}_{\alpha,\beta}$ in (7.6.48) follows with (7.5.34) and (7.2.10):

$$\begin{aligned} c^*\big((\alpha,0),(0,\beta)\big)\,\mathbf{B}_\beta\mathbf{A}_\alpha &= c^*\big((\alpha,0),(0,\beta)\big)\,\mathbf{D}_{0,\beta}\mathbf{D}_{\alpha,0} \\ &= c^*\big((\alpha,0),(0,\beta)\big)\,c\big((\alpha,0),(0,\beta)\big)\,\mathbf{D}_{\alpha,\beta} = \mathbf{D}_{\alpha,\beta}\,. \end{aligned}$$

We next show that $\mathbf{A}_\alpha$ is a faithful projective representation of $(\mathbb{R},+)$: We have $\mathbf{A}_0 = \mathbf{D}_{0,0} = \mathbf{I}$ and

$$\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{D}_{\alpha_2,0}\mathbf{D}_{\alpha_1,0} = c\big((\alpha_1,0),(\alpha_2,0)\big)\,\mathbf{D}_{\alpha_1+\alpha_2,0} = c_{\mathbf{A}}(\alpha_1,\alpha_2)\,\mathbf{A}_{\alpha_1+\alpha_2}$$

with $c_{\mathbf{A}}(\alpha_1,\alpha_2) = c\big((\alpha_1,0),(\alpha_2,0)\big)$. $\mathbf{A}_\alpha$ is faithful since $\mathbf{A}_\alpha = \mathbf{D}_{\alpha,0}$ and $\mathbf{D}_{\alpha,\beta}$ is faithful. The proof that $\mathbf{B}_\beta$ is a faithful projective representation of $(\mathbb{R},+)$ is analogous.

Finally, if $c\big((\alpha_1,\beta_1),(\alpha_2,\beta_2)\big) = e^{-j2\pi\alpha_2\beta_1}$, then $c\big((\alpha,0),(0,\beta)\big) = 1$ so that $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$. Furthermore, $c_{\mathbf{A}}(\alpha_1,\alpha_2) = c\big((\alpha_1,0),(\alpha_2,0)\big) = 1$ and similarly $c_{\mathbf{B}}(\beta_1,\beta_2) = 1$, and hence $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are unitary representations of $(\mathbb{R},+)$. We have

$$\begin{aligned} \mathbf{A}_\alpha\mathbf{B}_\beta &= \mathbf{D}_{\alpha,0}\mathbf{D}_{0,\beta} = c\big((0,\beta),(\alpha,0)\big)\,\mathbf{D}_{\alpha,\beta} = e^{-j2\pi\alpha\beta}\,\mathbf{D}_{\alpha,\beta} \\ &= e^{-j2\pi\alpha\beta}\,\mathbf{B}_\beta\mathbf{A}_\alpha\,, \end{aligned}$$

so that $\mathbf{B}_\beta\mathbf{A}_\alpha = e^{j2\pi\alpha\beta}\,\mathbf{A}_\alpha\mathbf{B}_\beta$, which shows that $\mathbf{B}_\beta$ is dual to $\mathbf{A}_\alpha$ (see Definition 7.5). □

Corollary 7.20 *Consider a group $(\mathcal{D},\circ)$ that is isomorphic to the affine group (i.e., the group $(\mathbb{R}^+\!\times\mathbb{R},\diamond)$ with $(\alpha_1,\beta_1)\diamond(\alpha_2,\beta_2) = (\alpha_1\alpha_2,\ \beta_1\alpha_2+\beta_2)$) with isomorphism $\psi:\mathcal{D}\to\mathbb{R}^+\!\times\mathbb{R}$. If $\{\mathbf{D}'_{\theta'}\}_{\theta'\in(\mathcal{D},\circ)}$ is a DO for $(\mathcal{D},\circ)$, with cocycle $c'\big((\alpha'_1,\beta'_1),(\alpha'_2,\beta'_2)\big)$, then $\mathbf{D}_\theta \triangleq \mathbf{D}'_{\psi^{-1}(\theta)}$ is a DO for the affine group, with cocycle $c\big((\alpha_1,\beta_1),(\alpha_2,\beta_2)\big) = c'\big(\psi^{-1}(\alpha_1,\beta_1),\psi^{-1}(\alpha_2,\beta_2)\big)$, and it is* separable *(up to a phase factor) according to*

$$\mathbf{D}_\theta \equiv \mathbf{D}_{\alpha,\beta} = c^*\big((\alpha,0),(1,\beta)\big)\,\mathbf{B}_\beta\mathbf{A}_\alpha \quad \text{with } \mathbf{A}_\alpha \triangleq \mathbf{D}_{\alpha,0}\,,\ \ \mathbf{B}_\beta \triangleq \mathbf{D}_{1,\beta}\,.$$

Here, $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are faithful projective representations of $(\mathbb{R}^+,\cdot)$ and $(\mathbb{R},+)$, respectively; they satisfy

$$\begin{aligned} &\mathbf{A}_1 = \mathbf{I}\,, &&\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = c_{\mathbf{A}}(\alpha_1,\alpha_2)\,\mathbf{A}_{\alpha_1\alpha_2} \quad \text{for all } \alpha_1,\alpha_2\in\mathbb{R}^+ \\ &\mathbf{B}_0 = \mathbf{I}\,, &&\mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} = c_{\mathbf{B}}(\beta_1,\beta_2)\,\mathbf{B}_{\beta_1+\beta_2} \quad \text{for all } \beta_1,\beta_2\in\mathbb{R}\,, \end{aligned}$$

with $c_{\mathbf{A}}(\alpha_1,\alpha_2) = c\big((\alpha_1,0),(\alpha_2,0)\big)$ and $c_{\mathbf{B}}(\beta_1,\beta_2) = c\big((1,\beta_1),(1,\beta_2)\big)$. If $c\big((\alpha_1,\beta_1),(\alpha_2,\beta_2)\big) \equiv 1$, then $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$, and $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are unitary representations of $(\mathbb{R}^+,\cdot)$ and $(\mathbb{R},+)$, respectively, with $\mathbf{B}_\beta$ affine to $\mathbf{A}_\alpha$.

Proof. The proof is essentially analogous to that of Corollary 7.19. □

The operator families $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ will be called *partial DOs* [26, 28, 30]. The last two corollaries state that a DO, after transforming the displacement parameter as $\theta' = \psi^{-1}(\theta)$, can be factored (possibly up to a phase factor) into the partial DOs which are projective or unitary representations of the 1-parameter groups $(\mathbb{R}, +)$ or $(\mathbb{R}^+, \cdot)$. (Note that $(\mathbb{R}^+, \cdot)$ is isomorphic to $(\mathbb{R}, +)$, so the respective partial DO could be reparameterized with $(\mathbb{R}, +)$.) This separable structure of DOs will be exploited in Section 7.7.

7.6.3 Unitary Equivalence Results

Two basic DOs are the operators $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ defined on $L_2(\mathbb{R})$ and $\mathbf{R}_{\sigma,\tau} = \mathbf{T}_\tau \mathbf{C}_\sigma$ defined on the Hardy space $H^2(\mathbb{R})$. Whereas $\mathbf{S}_{\tau,\nu}$ is a projective representation of the group $(\mathbb{R}^2, +)$, $\mathbf{R}_{\sigma,\tau}$ is a unitary representation of the affine group. In the previous subsection, we showed that all groups $(\mathcal{D}, \circ)$ underlying "proper" DOs (i.e., DOs for which there exists a DF) are isomorphic to $(\mathbb{R}^2, +)$ or the affine group. This suggests that proper DOs might be related to one of the basic DOs $\mathbf{S}_{\tau,\nu}$ and $\mathbf{R}_{\sigma,\tau}$. The next two theorems consider the two cases that are possible, namely, $(\mathcal{D}, \circ)$ isomorphic to $(\mathbb{R}^2, +)$ or the affine group.

Theorem 7.21 *Let $(\mathcal{D}, \circ)$ be isomorphic to $(\mathbb{R}^2, +)$ with isomorphism ψ. Let $\{\mathbf{D}'_{\theta'}\}_{\theta' \in (\mathcal{D}, \circ)}$ be a DO for $(\mathcal{D}, \circ)$. If the cocycle of $\mathbf{D}_\theta \triangleq \mathbf{D}'_{\psi^{-1}(\theta)}$ (recall from Corollary 7.19 that $\mathbf{D}_\theta$ is a DO for $(\mathbb{R}^2, +)$) is $c\big((\alpha_1, \beta_1), (\alpha_2, \beta_2)\big) = e^{-j2\pi\alpha_2\beta_1}$, then $\mathbf{D}_\theta$ is unitarily equivalent to $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$, i.e.,*

$$\mathbf{D}_{\alpha,\beta} = \mathbf{U}\mathbf{S}_{\alpha,\beta}\mathbf{U}^{-1} \,,$$

with $\mathbf{U}$ an invertible and isometric operator.

Conversely, let $\mathbf{D}'_{\theta'}$ with $\theta' \in (\mathcal{D}, \circ)$ be unitarily equivalent to $\mathbf{S}_{\tau,\nu}$ up to a phase factor and a parameter transformation by ψ, i.e.,

$$\mathbf{D}'_{\theta'} \;=\; e^{j\sigma(\psi(\theta'))}\,\mathbf{U}\mathbf{S}_{\psi(\theta')}\mathbf{U}^{-1} \,,$$

with $\mathbf{U}$ an invertible and isometric operator and $\sigma(\cdot)$ continuous and real-valued with $\sigma(0,0) = 0 \bmod 2\pi$. Then $\mathbf{D}'_{\theta'}$ is a DO for $(\mathcal{D}, \circ)$. Furthermore, the cocycle of $\mathbf{D}'_{\theta'}$ is given by $c'(\theta'_1, \theta'_2) = e^{j\phi(\theta'_1, \theta'_2)}$ with

$$\phi(\theta'_1, \theta'_2) \;=\; \sigma\big(\psi(\theta'_1)\big) + \sigma\big(\psi(\theta'_2)\big) - \sigma\big(\psi(\theta'_1 \circ \theta'_2)\big) - 2\pi\, p_2\big(\psi(\theta'_1)\big)\, p_1\big(\psi(\theta'_2)\big) \,, \tag{7.6.49}$$

where the "projection" mappings $p_i : \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ are defined as $p_1(\alpha, \beta) = \alpha$ and $p_2(\alpha, \beta) = \beta$.

Proof. Assume $(\mathcal{D}, \circ)$ is isomorphic to $(\mathbb{R}^2, +)$, $\mathbf{D}'_{\theta'}$ is a DO for $(\mathcal{D}, \circ)$, and the cocycle of $\mathbf{D}_\theta = \mathbf{D}'_{\psi^{-1}(\theta)}$ is $c\big((\alpha_1, \beta_1), (\alpha_2, \beta_2)\big) = e^{-j2\pi\alpha_2\beta_1}$. According to Corollary 7.19, $\mathbf{D}_\theta$ is a DO for $(\mathbb{R}^2, +)$, and $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta \mathbf{A}_\alpha$ with $\mathbf{B}_\beta$

dual to $\mathbf{A}_\alpha$. According to Definition 7.11, the underlying Hilbert space is minimal invariant under $\mathbf{D}_\theta$ and hence also under both $\mathbf{A}_\alpha = \mathbf{D}_{\alpha,0}$ and $\mathbf{B}_\beta = \mathbf{D}_{0,\beta}$. According to Theorem 7.7, if $\mathbf{B}_\beta$ is dual to $\mathbf{A}_\alpha$ on a minimal invariant Hilbert space, with $\alpha, \beta \in (\mathbb{R}, +)$, then there hold unitary equivalence relations $\mathbf{A}_\alpha = \mathbf{U}\mathbf{T}_\alpha\mathbf{U}^{-1}$ and $\mathbf{B}_\beta = \mathbf{U}\mathbf{F}_\beta\mathbf{U}^{-1}$; hence,

$$\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha = \mathbf{U}\mathbf{F}_\beta\mathbf{U}^{-1}\mathbf{U}\mathbf{T}_\alpha\mathbf{U}^{-1} = \mathbf{U}\mathbf{S}_{\alpha,\beta}\mathbf{U}^{-1} .$$

Conversely, assume that $\mathbf{D}'_{\theta'} = e^{j\sigma(\psi(\theta'))}\,\mathbf{U}\mathbf{S}_{\psi(\theta')}\mathbf{U}^{-1}$ with $\sigma(0,0) = 0 \mod 2\pi$. Then we have $\mathbf{D}'_{\theta'_0} = e^{j\sigma(0,0)}\,\mathbf{U}\mathbf{S}_{0,0}\mathbf{U}^{-1} = \mathbf{U}\mathbf{I}\mathbf{U}^{-1} = \mathbf{I}$ and

$$\begin{aligned}
\mathbf{D}'_{\theta'_2}\mathbf{D}'_{\theta'_1} &= e^{j\sigma(\psi(\theta'_2))}\,e^{j\sigma(\psi(\theta'_1))}\,\mathbf{U}\mathbf{S}_{\psi(\theta'_2)}\mathbf{U}^{-1}\mathbf{U}\mathbf{S}_{\psi(\theta'_1)}\mathbf{U}^{-1} \\
&= e^{j\left[\sigma(\psi(\theta'_2)) + \sigma(\psi(\theta'_1))\right]}\,e^{-j2\pi p_2(\psi(\theta'_1))\,p_1(\psi(\theta'_2))}\,\mathbf{U}\mathbf{S}_{\psi(\theta'_1\circ\theta'_2)}\mathbf{U}^{-1} \\
&= e^{j\left[\sigma(\psi(\theta'_1)) + \sigma(\psi(\theta'_2)) - 2\pi p_2(\psi(\theta'_1))\,p_1(\psi(\theta'_2))\right]}\,e^{-j\sigma(\psi(\theta'_1\circ\theta'_2))}\,\mathbf{D}'_{\theta'_1\circ\theta'_2} .
\end{aligned}$$

Thus, $\mathbf{D}'_{\theta'}$ is a projective representation of $(\mathcal{D},\circ)$ with cocycle phase (7.6.49). It is irreducible and faithful because $\mathbf{S}_{\alpha,\beta}$ is. Therefore, it is a DO. □

Theorem 7.22 *Let $(\mathcal{D},\circ)$ be isomorphic to the affine group with isomorphism ψ. Let $\{\mathbf{D}'_{\theta'}\}_{\theta'\in(\mathcal{D},\circ)}$ be a DO for $(\mathcal{D},\circ)$ with cocycle $c'(\theta'_1,\theta'_2) \equiv 1$. Then $\mathbf{D}_\theta \triangleq \mathbf{D}'_{\psi^{-1}(\theta)}$ is unitarily equivalent to $\mathbf{R}_{\sigma,\tau} = \mathbf{T}_\tau\mathbf{C}_\sigma$, i.e.,*

$$\mathbf{D}_{\alpha,\beta} = \mathbf{V}\mathbf{R}_{\alpha,\pm\beta}\mathbf{V}^{-1} ,$$

with $\mathbf{V}$ an invertible and isometric operator.

Conversely, let $\mathbf{D}'_{\theta'}$ with $\theta' \in (\mathcal{D},\circ)$ be unitarily equivalent to $\mathbf{R}_{\sigma,\tau}$ up to a parameter transformation, i.e.,

$$\mathbf{D}'_{\theta'} = \mathbf{V}\mathbf{R}_{\psi(\theta')}\mathbf{V}^{-1} ,$$

with $\mathbf{V}$ an invertible and isometric operator. Then $\mathbf{D}'_{\theta'}$ is a DO for $(\mathcal{D},\circ)$, with cocycle $c'(\theta'_1,\theta'_2) \equiv 1$.

Proof. The proof is essentially analogous to that of Theorem 7.21. □

7.6.4 Relation between Extended Displacement Functions

Next, we show an important relation between different extended DFs based on the same group.

Theorem 7.23 *Let $\mathcal{Z}$ and $\hat{\mathcal{Z}}$ be open, simply connected subsets of $\mathbb{R}^2$, $(\mathcal{D},\circ)$ a simply connected 2-D Lie group, and $e : \mathcal{Z}\times\mathcal{D} \to \mathcal{Z}$ an extended DF for $(\mathcal{D},\circ)$ acting on $\mathcal{Z}$. If $v : \mathcal{Z} \to \hat{\mathcal{Z}}$ is a diffeomorphism (i.e., an invertible, continuously differentiable function that maps $\mathcal{Z}$ onto $\hat{\mathcal{Z}}$ with nonzero Jacobian) with constant Jacobian, then the function*

$$\hat{e} : \hat{\mathcal{Z}}\times\mathcal{D} \to \hat{\mathcal{Z}} , \qquad \hat{e}(\hat{z},\theta) = v\big(e\big(v^{-1}(\hat{z}),\theta\big)\big) \qquad (7.6.50)$$

is an extended DF for $(\mathcal{D}, \circ)$ *acting on* $\hat{\mathcal{Z}}$.

Conversely, let $e : \mathcal{Z} \times \mathcal{D} \to \mathcal{Z}$ *and* $\hat{e} : \hat{\mathcal{Z}} \times \mathcal{D} \to \hat{\mathcal{Z}}$ *be extended DFs for* $(\mathcal{D}, \circ)$ *acting on* $\mathcal{Z}$ *and* $\hat{\mathcal{Z}}$, *respectively. Then* e *and* $\hat{e}$ *are related according to (7.6.50) with* $v : \mathcal{Z} \to \hat{\mathcal{Z}}$ *given by*

$$v(z) = \hat{d}\big(d^{-1}(z)\big)\,, \tag{7.6.51}$$

where $d(\theta) = e(z_0, \theta)$ *and* $\hat{d}(\theta) = \hat{e}(\hat{z}_0, \theta)$ *with arbitrary but fixed* $z_0 \in \mathcal{Z}$ *and* $\hat{z}_0 \in \hat{\mathcal{Z}}$. *This function* v *is a diffeomorphism with constant Jacobian.*

Proof. We first show that $\hat{e}(\hat{z}, \theta)$ in (7.6.50) satisfies the axioms of Definition 7.16. Axiom 1 is satisfied since a composition of invertible and continuously differentiable functions is invertible and continuously differentiable. Furthermore, $\hat{e}$ is area-preserving since (J denotes the Jacobian; furthermore note that $J_v(z)$ is constant by assumption and $J_{e(\cdot,\theta)}(z) = \pm 1$ due to area preservation)

$$J_{\hat{e}(\cdot,\theta)}(\hat{z}) \;=\; \frac{J_v\big(e\big(v^{-1}(\hat{z}),\,\theta\big)\big)\,J_{e(\cdot,\theta)}\big(v^{-1}(\hat{z})\big)}{J_v\big(v^{-1}(\hat{z})\big)} \;=\; J_{e(\cdot,\theta)}\big(v^{-1}(\hat{z})\big) \;=\; \pm 1\,.$$

Axiom 2 is satisfied since a composition of diffeomorphisms is a diffeomorphism. Axiom 3, the composition property (7.6.45), can be verified in a straightforward manner. Hence, $\hat{e}(\hat{z}, \theta)$ is an extended DF.

Next, we prove the converse statement. As a composition of diffeomorphisms (see Theorem 7.17), v in (7.6.51) is a diffeomorphism. Inserting $\hat{e}(\hat{z}, \theta) = \hat{d}(\hat{d}^{-1}(\hat{z}) \circ \theta)$ (see (7.6.46)) into the left-hand side of (7.6.50), as well as (7.6.51) and $e(z, \theta) = d\big(d^{-1}(z) \circ \theta\big)$ into the right-hand side of (7.6.50), we see that the two sides are equal and, hence, that (7.6.50) is true. Finally, setting $\hat{z} = v(z_1)$ in (7.6.50), with $z_1 \in \mathcal{Z}$ arbitrary but fixed, we obtain $\hat{e}(v(z_1), \theta) = v\big(e(z_1, \theta)\big)$ and hence $J_{\hat{e}(\cdot,\theta)}(v(z_1))\, J_v(z_1) = J_v(e(z_1, \theta))\, J_{e(\cdot,\theta)}(z_1)$. Since $J_{\hat{e}} = \pm 1$ and $J_e = \pm 1$ due to area preservation (see Axiom 1 in Definition 7.16), this yields $J_v(z_1) = \pm J_v(e(z_1, \theta))$. This equation holds for all $\theta \in \mathcal{D}$. Since to every $z \in \mathcal{Z}$ we can find a $\theta \in \mathcal{D}$ such that $e(z_1, \theta) = z$ (see Axiom 2 in Definition 7.16), we get $J_v(z) = \pm J_v(z_1)$ for all $z \in \mathcal{Z}$. But since $J_v(z)$ is continuous and $\mathcal{Z}$ is connected, there is in fact $J_v(z) = J_v(z_1)$ for all $z \in \mathcal{Z}$, i.e., $J_v(z)$ is constant. □

Hence, the extended DFs of different DOs for the *same* group are related through diffeomorphisms v. (Note that the function v in Theorem 7.23 is not unique since z_0 and $\hat{z}_0$ are arbitrary.)

For $(\mathbb{R}^2, +)$ and the affine group (and all isomorphic groups), Kirillov's coadjoint orbit theory [53] allows the construction of a unique function e, called *coadjoint action* of the group, that satisfies the DF axioms in Definition 7.16. This function acts on sets called *coadjoint orbits*, some of which can be identified with open, simply connected subsets $\mathcal{Z} \subseteq \mathbb{R}^2$ [15, 17, 53, 54, 56]. Thus, the coadjoint orbit theory allows the construction of a *particular* extended DF for $(\mathbb{R}^2, +)$ and the affine group. It has been conjectured [12] that extended DFs are generally identical to the coadjoint actions. However, the coadjoint action of a given group is unique whereas there exist different extended DFs for this group (the extended DFs of different DOs based on this group, see Theorem 7.23). In fact, an extended DF is associated to a specific DO $\mathbf{D}_\theta$, and not generically to the underlying group $(\mathcal{D}, \circ)$. Thus, in general the coadjoint orbit theory does *not* allow the construction of the extended DF of a given DO.

7.7 The Displacement Function: Construction

We shall now present a systematic method for constructing the DF of a given DO. This method avoids certain theoretical inconveniences of previous approaches [25, 28].

7.7.1 General Construction

Consider a DO $\{\mathbf{D}'_{\theta'}\}_{\theta' \in (\mathcal{D}, \circ)}$ whose DF is assumed to exist. According to Corollaries 7.19 and 7.20, the associated DO $\mathbf{D}_\theta \triangleq \mathbf{D}'_{\psi^{-1}(\theta)}$ is separable, i.e., $\mathbf{D}_{\alpha,\beta} = e^{j\sigma(\alpha,\beta)}\, \mathbf{B}_\beta \mathbf{A}_\alpha$, where the *partial DOs* $\{\mathbf{A}_\alpha\}_{\alpha \in (\mathcal{A}, \bullet)}$ and $\{\mathbf{B}_\beta\}_{\beta \in (\mathcal{B}, *)}$ are projective representations of groups $(\mathcal{A}, \bullet)$ and $(\mathcal{B}, *)$ that are $(\mathbb{R}, +)$ or isomorphic to $(\mathbb{R}, +)$. This separability will allow us to construct the extended DF of $\mathbf{D}_{\alpha,\beta}$ by composing the individual extended DFs of the partial DOs $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$. The extended DF of $\mathbf{D}'_{\theta'}$ can finally be obtained by a reparameterization of the extended DF of $\mathbf{D}_\theta$ using $\theta = \psi(\theta')$.

We first consider $\mathbf{A}_\alpha$. If a signal $x(t)$ is localized about some TF point $z_1 = (t_1, f_1)$, then $(\mathbf{A}_\alpha x)(t)$ will be localized about some other TF point $z_2 = e_{\mathbf{A}}(z_1, \alpha)$. We shall call the function $e_{\mathbf{A}} : \mathcal{Z} \times \mathcal{A} \to \mathcal{Z}$ the extended DF of the partial DO $\mathbf{A}_\alpha$, even though Axiom 2 in Definition 7.16 is not satisfied. To find $z_2 = (t_2, f_2)$, we use the (generalized) eigenfunctions $u_b^{\mathbf{A}}(t)$ of $\mathbf{A}_\alpha$ that are defined by the eigenvalue relation

$$\big(\mathbf{A}_\alpha u_b^{\mathbf{A}}\big)(t) \,=\, \lambda^{\mathbf{A}}_{\alpha,b}\, u_b^{\mathbf{A}}(t)\,, \qquad \alpha \in (\mathcal{A}, \bullet)\,,\ b \in (\tilde{\mathcal{A}}, \tilde{\bullet})\,. \tag{7.7.52}$$

Here, the eigenfunctions $u_b^{\mathbf{A}}(t)$ are indexed by $b \in (\tilde{\mathcal{A}}, \tilde{\bullet})$, with $(\tilde{\mathcal{A}}, \tilde{\bullet})$ again isomorphic to $(\mathbb{R}, +)$ [36].

The TF locus of $u_b^{\mathbf{A}}(t)$ is defined by the instantaneous frequency $\nu\{u_b^{\mathbf{A}}\}(t)$ or the group delay $\tau\{u_b^{\mathbf{A}}\}(f)$, whichever exists. (The instantaneous frequency of a signal $x(t)$ is $\nu\{x\}(t) = \frac{1}{2\pi}\frac{d}{dt}\arg\{x(t)\}$; it exists if $\arg\{x(t)\}$ is differentiable and $x(t) \neq 0$ almost everywhere. The group delay of $x(t)$ is $\tau\{x\}(f) = -\frac{1}{2\pi}\frac{d}{df}\arg\{X(f)\}$ with $X(f) = \int_t x(t)\, e^{-j2\pi f t} dt$; it exists if $\arg\{X(f)\}$ is differentiable and $X(f) \neq 0$ almost everywhere.) Here, e.g., we shall assume existence of $\nu\{u_b^{\mathbf{A}}\}(t)$. Let us choose b_1 such that the TF

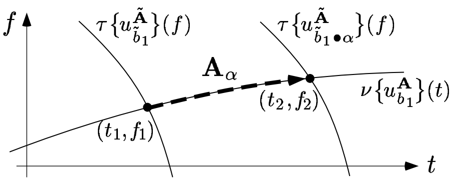

FIGURE 7.1. *Construction of the extended DF of* $\mathbf{A}_\alpha$.

curve defined by $\nu\{u_{b_1}^{\mathbf{A}}\}(t)$ passes through (t_1, f_1), i.e.,

$$\nu\{u_{b_1}^{\mathbf{A}}\}(t_1) = f_1 . \tag{7.7.53}$$

This is shown in Fig. 7.1. Since the instantaneous frequency is invariant to constant factors of the signal, (7.7.52) implies

$$\nu\{\mathbf{A}_\alpha u_{b_1}^{\mathbf{A}}\}(t) = \nu\{u_{b_1}^{\mathbf{A}}\}(t) .$$

This shows that $\mathbf{A}_\alpha$ preserves the TF locus of $u_{b_1}^{\mathbf{A}}(t)$, in the sense that *all TF points on the curve* $\nu\{u_{b_1}^{\mathbf{A}}\}(t)$*—including* (t_1, f_1)*—are mapped again onto TF points on* $\nu\{u_{b_1}^{\mathbf{A}}\}(t)$. Hence, $(t_2, f_2) = e_{\mathbf{A}}\big((t_1, f_1), \alpha\big)$ must lie on $\nu\{u_{b_1}^{\mathbf{A}}\}(t)$ (see Fig. 7.1), i.e., there must be

$$\nu\{u_{b_1}^{\mathbf{A}}\}(t_2) = f_2 . \tag{7.7.54}$$

To find the exact position of (t_2, f_2) on the TF curve defined by $\nu\{u_{b_1}^{\mathbf{A}}\}(t)$, we consider the operator $\{\tilde{\mathbf{A}}_{\tilde{\alpha}}\}_{\tilde{\alpha} \in (\tilde{\mathcal{A}}, \tilde{\bullet})}$ that is dual to $\{\mathbf{A}_\alpha\}_{\alpha \in (\mathcal{A}, \bullet)}$ (see Definition 7.5). We note that the (generalized) eigenfunctions $\{u_{\tilde{b}}^{\tilde{\mathbf{A}}}(t)\}_{\tilde{b} \in (\mathcal{A}, \bullet)}$ of $\tilde{\mathbf{A}}_{\tilde{\alpha}}$ can be derived from the eigenfunctions $\{u_b^{\mathbf{A}}(t)\}_{b \in (\tilde{\mathcal{A}}, \tilde{\bullet})}$ of $\mathbf{A}_\alpha$ as [30]

$$u_{\tilde{b}}^{\tilde{\mathbf{A}}}(t) = \int_{\tilde{\mathcal{A}}} u_b^{\mathbf{A}}(t)\, e^{-j2\pi \psi_{\mathcal{A}}(\tilde{b}) \psi_{\tilde{\mathcal{A}}}(b)}\, d\psi_{\tilde{\mathcal{A}}}(b) .$$

(Note also that $\tilde{\mathbf{A}}_{\tilde{\alpha}}$ and $u_{\tilde{b}}^{\tilde{\mathbf{A}}}(t)$ are assumed to be indexed by the group $(\tilde{\mathcal{A}}, \tilde{\bullet})$ and $(\mathcal{A}, \bullet)$, respectively; such an indexing is always possible since all these groups are isomorphic to $(\mathbb{R}, +)$, and hence isomorphic to each other.) The TF locus of $u_{\tilde{b}}^{\tilde{\mathbf{A}}}(t)$ is defined by the instantaneous frequency $\nu\{u_{\tilde{b}}^{\tilde{\mathbf{A}}}\}(t)$ or the group delay $\tau\{u_{\tilde{b}}^{\tilde{\mathbf{A}}}\}(f)$, whichever exists. Here, e.g., we assume existence of $\tau\{u_{\tilde{b}}^{\tilde{\mathbf{A}}}\}(f)$. Let us choose $\tilde{b}_1$ such that the TF curve defined by $\tau\{u_{\tilde{b}_1}^{\tilde{\mathbf{A}}}\}(f)$ passes through (t_1, f_1), i.e.,

$$\tau\{u_{\tilde{b}_1}^{\tilde{\mathbf{A}}}\}(f_1) = t_1 , \tag{7.7.55}$$

as shown in Fig. 7.1. According to (7.3.27), the eigenfunctions $u_{\tilde{b}}^{\tilde{\mathbf{A}}}(t)$ (suitably parameterized) satisfy

$$\big(\mathbf{A}_\alpha u_{\tilde{b}}^{\tilde{\mathbf{A}}}\big)(t) = u_{\tilde{b} \bullet \alpha}^{\tilde{\mathbf{A}}}(t) , \qquad \alpha, \tilde{b} \in (\mathcal{A}, \bullet) , \tag{7.7.56}$$

and hence there must be

$$\tau\{\mathbf{A}_\alpha u^{\tilde{\mathbf{A}}}_{\tilde{b}_1}\}(f) \,=\, \tau\{u^{\tilde{\mathbf{A}}}_{\tilde{b}_1 \bullet \alpha}\}(f)\,.$$

This shows that $\mathbf{A}_\alpha$ *maps all TF points on* $\tau\{u^{\tilde{\mathbf{A}}}_{\tilde{b}_1}\}(f)$*—including* (t_1, f_1)*— onto TF points on* $\tau\{u^{\tilde{\mathbf{A}}}_{\tilde{b}_1 \bullet \alpha}\}(f)$. Hence, $(t_2, f_2) = e_{\mathbf{A}}\big((t_1, f_1), \alpha\big)$ must lie on $\tau\{u^{\tilde{\mathbf{A}}}_{\tilde{b}_1 \bullet \alpha}\}(f)$ (see Fig. 7.1), i.e., there must be

$$\tau\{u^{\tilde{\mathbf{A}}}_{\tilde{b}_1 \bullet \alpha}\}(f_2) \,=\, t_2\,. \tag{7.7.57}$$

The relations (7.7.54) and (7.7.57) constitute a (generally nonlinear) system of equations in t_2 and f_2 whose solution for given $b_1, \tilde{b}_1$ (which follow from t_1, f_1 according to (7.7.53) and (7.7.55), respectively) and for given α defines the extended DF $e_{\mathbf{A}}$ via the identity $(t_2, f_2) \equiv e_{\mathbf{A}}\big((t_1, f_1), \alpha\big)$, provided that this solution exists and is unique. The construction of $e_{\mathbf{A}}$ can now be summarized as follows (see Fig. 7.1):

1. For any given $(t_1, f_1) \in \mathcal{Z}$, we calculate associated eigenfunction parameters $b_1 \in (\tilde{\mathcal{A}}, \tilde{\bullet})$, $\tilde{b}_1 \in (\mathcal{A}, \bullet)$ as the solution to the equations (see (7.7.53), (7.7.55))

$$\nu\{u^{\mathbf{A}}_{b_1}\}(t_1) \,=\, f_1\,, \qquad \tau\{u^{\tilde{\mathbf{A}}}_{\tilde{b}_1}\}(f_1) \,=\, t_1\,. \tag{7.7.58}$$

2. The extended DF $e_{\mathbf{A}}$ is defined by $(t_2, f_2) \equiv e_{\mathbf{A}}\big((t_1, f_1), \alpha\big)$, where (t_2, f_2) is obtained as the solution to the system of equations (see (7.7.54), (7.7.57))

$$\nu\{u^{\mathbf{A}}_{b_1}\}(t_2) \,=\, f_2\,, \qquad \tau\{u^{\tilde{\mathbf{A}}}_{\tilde{b}_1 \bullet \alpha}\}(f_2) \,=\, t_2\,. \tag{7.7.59}$$

This construction is easily modified for the case where, e.g., $\tau\{u^{\mathbf{A}}_{b}\}(f)$ and $\nu\{u^{\tilde{\mathbf{A}}}_{\tilde{b}}\}(t)$ exist instead of $\nu\{u^{\mathbf{A}}_{b}\}(t)$ and $\tau\{u^{\tilde{\mathbf{A}}}_{\tilde{b}}\}(f)$.

The extended DF $e_{\mathbf{B}}(z, \beta)$ of $\mathbf{B}_\beta$ can be constructed by an analogous procedure, using the eigenfunctions $\{u^{\mathbf{B}}_a(t)\}_{a \in (\tilde{\mathcal{B}}, \tilde{*})}$ and $\{u^{\tilde{\mathbf{B}}}_{\tilde{a}}(t)\}_{\tilde{a} \in (\mathcal{B}, *)}$ of $\{\mathbf{B}_\beta\}_{\beta \in (\mathcal{B}, *)}$ and of the dual operator $\{\tilde{\mathbf{B}}_{\tilde{\beta}}\}_{\tilde{\beta} \in (\tilde{\mathcal{B}}, \tilde{*})}$, respectively. Next, the extended DF $e(z, \theta)$ of the DO $\mathbf{D}_{\alpha,\beta} = e^{j\sigma(\alpha,\beta)}\,\mathbf{B}_\beta \mathbf{A}_\alpha$ follows upon composing $e_{\mathbf{A}}(z, \alpha)$ and $e_{\mathbf{B}}(z, \beta)$,

$$e\big((t, f), (\alpha, \beta)\big) \,=\, e_{\mathbf{B}}\big(e_{\mathbf{A}}\big((t, f), \alpha\big),\, \beta\big)\,. \tag{7.7.60}$$

Finally, the extended DF $e'(z, \theta')$ of $\mathbf{D}'_{\theta'}$ is obtained by reparameterizing $e(z, \theta)$ according to $e'(z, \theta') \,\triangleq\, e\,(z, \psi(\theta'))$. The DF of $\mathbf{D}'_{\theta'}$ is given by $d'(z) = e'(z_0, \theta')$ with some $z_0 \in \mathcal{Z}$.

This construction of the (extended) DF assumes that the instantaneous frequencies or group delays of the operators $\mathbf{A}_\alpha$, $\tilde{\mathbf{A}}_{\tilde{\alpha}}$, $\mathbf{B}_\beta$, and $\tilde{\mathbf{B}}_{\tilde{\beta}}$ exist and that the equations (7.7.58) and (7.7.59) have unique solutions. While the construction is intuitively reasonable, there is no general proof that the resulting function $e'(z, \theta')$ satisfies the axioms of Definition 7.16. However, we will show in Subsections 7.7.3 and 7.7.4 that for the important and fairly extensive cases of dual and affine modulation and warping operators, all assumptions and axioms are satisfied and the extended DF is readily obtained in closed form.

7.7.2 Warping Operator and Modulation Operator

By way of example and for later use, let us calculate the extended DF of a *warping operator* $\{\mathbf{W}_\alpha\}_{\alpha \in (\mathcal{A}, \bullet)}$ with warping function $w_\alpha(t)$ defined on a set $\Theta \subseteq \mathbb{R}$ (see Definition 7.2). According to the second part of Theorem 7.3, under specific assumptions there exists a subset $\Omega \subseteq \Theta$ such that

$$w_\alpha(t) = n^{-1}\big(n(t) \bullet \alpha^{-1}\big), \qquad t \in \Omega, \tag{7.7.61}$$

with an invertible function $n : \Omega \to \mathcal{A}$. We recall from (7.2.21) that the eigenfunctions of $\mathbf{W}_\alpha$ are given by $u_b^{\mathbf{W}}(t) = \sqrt{|\tilde{n}'(t)|}\, e^{j2\pi\, \psi_{\tilde{\mathcal{A}}}(b)\, \tilde{n}(t)}$, with $b \in \tilde{\mathcal{A}}$ and $\tilde{n}(t) = \psi_{\mathcal{A}}(n(t))$, where $\tilde{\mathcal{A}}$ is some group isomorphic to $(\mathbb{R}, +)$. The instantaneous frequency of $u_b^{\mathbf{W}}(t)$ is obtained as $\nu\{u_b^{\mathbf{W}}\}(t) = \psi_{\tilde{\mathcal{A}}}(b)\, \tilde{n}'(t)$. According to Subsection 7.3.2, the dual operator $\{\widetilde{\mathbf{W}}_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \tilde{\mathcal{A}}}$ is the modulation operator with modulation function $n(t)$. We recall from (7.2.15) that the eigenfunctions of $\widetilde{\mathbf{W}}_{\tilde{\alpha}}$ are given by $u_{\tilde{b}}^{\widetilde{\mathbf{W}}}(t) = r(t)\, \delta\big(\tilde{n}(t) - \psi_{\mathcal{A}}(\tilde{b})\big)$, $\tilde{b} \in \mathcal{A}$ (assuming parameterization by the group $(\mathcal{A}, \bullet)$). The group delay of $u_{\tilde{b}}^{\widetilde{\mathbf{W}}}(t)$ is obtained as $\tau\{u_{\tilde{b}}^{\widetilde{\mathbf{W}}}\}(f) \equiv n^{-1}(\tilde{b})$. Evaluating (7.7.58) yields $\tilde{b}_1 = n(t_1)$ and $b_1 = \psi_{\tilde{\mathcal{A}}}^{-1}\big(f_1/\tilde{n}'(t_1)\big)$. Inserting into (7.7.59) and solving for t_2 and f_2 yields $t_2 = w_{\alpha^{-1}}(t_1)$ and $f_2 = f_1/w'_{\alpha^{-1}}(t_1)$, and hence the extended DF of $\mathbf{W}_\alpha$ is obtained as

$$e_{\mathbf{W}}\big((t,f), \alpha\big) = \left(w_{\alpha^{-1}}(t),\, \frac{f}{w'_{\alpha^{-1}}(t)}\right), \qquad (t,f) \in \Omega \times \mathbb{R}, \;\; \alpha \in \mathcal{A}. \tag{7.7.62}$$

Next, we calculate the extended DF of a *modulation operator* $\{\mathbf{M}_\beta\}_{\beta \in (\mathcal{B}, *)}$ with modulation function $m : \Omega \to \tilde{\mathcal{B}}$ defined on a set $\Omega \subseteq \mathbb{R}$ (see Definition 7.1). The group delay of the eigenfunctions of $\mathbf{M}_\beta$ is given by (cf. the previous calculation for $\widetilde{\mathbf{W}}_{\tilde{\alpha}}$) $\tau\{u_a^{\mathbf{M}}\}(f) \equiv m^{-1}(a)$, $a \in (\tilde{\mathcal{B}}, \tilde{*})$. The dual operator $\{\widetilde{\mathbf{M}}_{\tilde{\beta}}\}_{\tilde{\beta} \in \tilde{\mathcal{B}}}$ can be shown to be the warping operator with warping function $w_{\tilde{\beta}}(t) = m^{-1}\big(m(t) \,\tilde{*}\, \tilde{\beta}\big)$. Assuming parameterization of the eigenfunctions of $\widetilde{\mathbf{M}}_{\tilde{\beta}}$ according to $u_{\tilde{a}}^{\widetilde{\mathbf{M}}}(t) = \sqrt{|\tilde{m}'(t)|}\, e^{j2\pi\, \psi_{\mathcal{B}}(\tilde{a})\, \tilde{m}(t)}$ (cf. (7.2.21)), with $\tilde{a} \in (\mathcal{B}, *)$ and $\tilde{m}(t) = \psi_{\tilde{\mathcal{B}}}\big(m(t)\big)$, the instantaneous frequency of $u_{\tilde{a}}^{\widetilde{\mathbf{M}}}(t)$ is given by $\nu\{u_{\tilde{a}}^{\widetilde{\mathbf{M}}}\}(t) = \psi_{\mathcal{B}}(\tilde{a})\, \tilde{m}'(t)$. Evaluating (7.7.58) (with the roles of instantaneous frequency and group delay interchanged) yields $a_1 = m(t_1)$ and $\tilde{a}_1 = \psi_{\mathcal{B}}^{-1}\big(\frac{f_1}{\tilde{m}'(t_1)}\big)$. Inserting into (7.7.59) (again with the roles of instantaneous frequency and group delay interchanged) and solving for (t_2, f_2), we obtain the extended DF of $\mathbf{M}_\beta$ as

$$e_{\mathbf{M}}\big((t,f),\beta\big) = \big(t,\, f + \psi_{\mathcal{B}}(\beta)\tilde{m}'(t)\big)\,, \qquad (t,f) \in \Omega \times \mathbb{R}, \;\; \beta \in \mathcal{B}\,. \quad (7.7.63)$$

7.7.3 Dual Modulation and Warping Operators

We now consider $\mathbf{D}_{\alpha,\beta} = \mathbf{M}_\beta \mathbf{W}_\alpha$, where $\{\mathbf{M}_\beta\}_{\beta \in (\mathcal{B},*)}$ is a modulation operator with modulation function $m : \Omega \to \mathcal{A}$, $m(\Omega) = \mathcal{A}$ and $\{\mathbf{W}_\alpha\}_{\alpha \in (\mathcal{A},\bullet)}$ is the warping operator with warping function $w_\alpha(t) = m^{-1}(m(t) \bullet \alpha^{-1})$. According to Subsection 7.3.2, $\mathbf{M}_\beta$ is *dual* to $\mathbf{W}_\alpha$. Hence, due to Theorem 7.6, $\mathbf{D}_{\alpha,\beta}$ is a DO for a group $(\mathcal{A} \times \mathcal{B}, \circ)$ that is isomorphic to $(\mathbb{R}^2, +)$. Composing the extended DFs $e_{\mathbf{W}}\big((t,f),\alpha\big)$ in (7.7.62) and $e_{\mathbf{M}}\big((t,f),\beta\big)$ in (7.7.63) according to (7.7.60), the extended DF of $\mathbf{D}_{\alpha,\beta}$ is obtained as

$$e\big((t,f),(\alpha,\beta)\big) \;=\; e_{\mathbf{M}}\big(e_{\mathbf{W}}\big((t,f),\alpha\big),\, \beta\big) \;=\; \left(w_{\alpha^{-1}}(t)\,,\; \frac{f + \psi_{\mathcal{B}}(\beta)\tilde{m}'(t)}{w'_{\alpha^{-1}}(t)}\right), \quad (7.7.64)$$

with $(t,f) \in \Omega \times \mathbb{R}$, $(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}$, $\tilde{m}(t) = \psi_{\mathcal{A}}\big(m(t)\big)$, and $w_\alpha(t) = m^{-1}(m(t) \bullet \alpha^{-1})$. Setting $(t,f) = (t_0, f_0)$ with $t_0 = \tilde{m}^{-1}(0)$ and $f_0 = 0$ for simplicity, the DF of $\mathbf{D}_{\alpha,\beta}$ results as

$$d(\alpha,\beta) \;=\; \big(m^{-1}(\alpha)\,,\, \psi_{\mathcal{B}}(\beta)\,\tilde{m}'(m^{-1}(\alpha))\big)\,. \qquad (7.7.65)$$

It can be verified that the extended DF satisfies all axioms of Definition 7.16, provided that $\tilde{m}(t)$ is twice continuously differentiable and $\psi_{\mathcal{A}}(\alpha)$ and $\psi_{\mathcal{B}}(\beta)$ are continuously differentiable.

As an example, let $\mathbf{M}_\nu = \mathbf{F}_\nu$ and $\mathbf{W}_\tau = \mathbf{T}_\tau$, so that $\mathbf{D}_{\tau,\nu} = \mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$. Here, $m(t) = \tilde{m}(t) = t$, $w_\tau(t) = t - \tau$, $\Omega = \mathbb{R}$, and $(\mathcal{A},\bullet) = (\mathcal{B},*) = (\mathbb{R},+)$, so that (7.7.64) and (7.7.65) become

$$e\big((t,f),(\tau,\nu)\big) = (t+\tau, f+\nu)\,, \qquad d(\tau,\nu) = (\tau,\nu)\,, \qquad t,f,\tau,\nu \in \mathbb{R}\,.$$

7.7.4 Affine Modulation and Warping Operators

Next, let $\{\mathbf{M}_\beta\}_{\beta \in (\mathcal{B},*)}$ be a modulation operator with modulation function $m : \Omega \to \mathcal{A}$, with $\tilde{m}(\Omega) = \mathbb{R}^+$ or $\tilde{m}(\Omega) = \mathbb{R}^-$ where $\tilde{m}(t) = \psi_{\mathcal{A}}\big(m(t)\big)$. Furthermore, let $\{\mathbf{W}_\alpha\}_{\alpha \in (\mathcal{A},\bullet)}$ be the warping operator whose warping function is related to $m(t)$ according to (7.4.32), i.e., $w_\alpha(t) = \tilde{m}^{-1}\big(\tilde{m}(t)\exp(\psi_{\mathcal{A}}(\alpha))\big)$. The construction of the extended DF of $\mathbf{W}_\alpha$ requires the dual operator of $\mathbf{W}_\alpha$, which is a modulation operator with modulation function $n(t)$ related to $w_\alpha(t)$ according to $w_\alpha(t) = n^{-1}\big(n(t) \bullet \alpha^{-1}\big)$ (see (7.7.61)). Hence, hereafter we restrict all functions and operators to a subset $\Omega' \subseteq \Omega$ on which $w_\alpha(t)$ can be represented in the form $w_\alpha(t) = n^{-1}\big(n(t) \bullet \alpha^{-1}\big)$ with an invertible function $n : \Omega' \to \mathcal{A}$.

Consider $\mathbf{D}_{\alpha,\beta} = \mathbf{M}_\beta \mathbf{W}_\alpha$. With $w_\alpha(t) = \tilde{m}^{-1}\big(\tilde{m}(t)\exp(\psi_{\mathcal{A}}(\alpha))\big)$, it follows from Subsection 7.4.2 that $\mathbf{M}_\beta$ is *affine* to $\mathbf{W}_\alpha$. Hence, due to Theorem 7.9, $\mathbf{D}_{\alpha,\beta}$ is a DO for a group $(\mathcal{A} \times \mathcal{B}, \circ)$ that is isomorphic to the affine group. Using (7.7.62) and (7.7.63), the extended DF of $\mathbf{D}_{\alpha,\beta}$ is obtained as

$$\begin{aligned} e\big((t,f),(\alpha,\beta)\big) &= e_{\mathbf{M}}\big(e_{\mathbf{W}}\big((t,f),\alpha\big),\,\beta\big) \\ &= \left(w_{\alpha^{-1}}(t)\,,\; \frac{f + \psi_{\mathcal{B}}(\beta)\,\tilde{m}'(t)\exp\big(-\psi_{\mathcal{A}}(\alpha)\big)}{w'_{\alpha^{-1}}(t)} \right), \end{aligned} \tag{7.7.66}$$

with $(t,f) \in \Omega' \times \mathbb{R}$, $(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}$, $\tilde{m}(t) = \psi_{\mathcal{A}}\big(m(t)\big)$, and $w_\alpha(t) = \tilde{m}^{-1}\big(\tilde{m}(t)\exp(\psi_{\mathcal{A}}(\alpha))\big)$. Setting $(t,f) = (t_0, f_0)$ with $t_0 = \tilde{m}^{-1}(\pm 1)$ (the sign depends on whether $\tilde{m}(\Omega) = \mathbb{R}^+$ or $\tilde{m}(\Omega) = \mathbb{R}^-$) and $f_0 = 0$ for simplicity, the DF of $\mathbf{D}_{\alpha,\beta}$ results as

$$d(\alpha,\beta) = \big(\xi(\alpha)\,,\,\psi_{\mathcal{B}}(\beta)\,\tilde{m}'\big(\xi(\alpha)\big)\big), \quad \text{with } \xi(\alpha) \triangleq \tilde{m}^{-1}\big(\pm\exp(-\psi_{\mathcal{A}}(\alpha))\big). \tag{7.7.67}$$

Again, it can be verified that the extended DF satisfies all axioms of Definition 7.16, provided that $\tilde{m}(t)$ is twice continuously differentiable and $\psi_{\mathcal{A}}(\alpha)$ and $\psi_{\mathcal{B}}(\beta)$ are continuously differentiable.

As an example, let $\mathbf{M}_\nu = \mathbf{F}_\nu$ and $\mathbf{W}_\sigma = \mathbf{C}_\sigma$ with $\sigma \in \mathbb{R}^+$, i.e., $\mathbf{D}_{\sigma,\nu} = \mathbf{F}_\nu \mathbf{C}_\sigma$. Here, $\tilde{m}(t) = t$, $w_\sigma(t) = t/\sigma$, $n(t) = t$, $\psi_{\mathcal{A}}(\sigma) = -\ln\sigma$, $\psi_{\mathcal{B}}(\nu) = \nu$, and $\Omega = \Omega' = \mathbb{R}^+$, so that (7.7.66) and (7.7.67) become

$$e\big((t,f),(\sigma,\nu)\big) = \left(\sigma t, \frac{f}{\sigma} + \nu\right), \qquad d(\sigma,\nu) = (\sigma,\nu)\,, \qquad t,\sigma \in \mathbb{R}^+,\; f,\nu \in \mathbb{R}.$$

7.8 Covariant Signal Representations: Time-Frequency Domain

In Section 7.5, we derived covariant linear and bilinear signal representations that were functions of the displacement parameter $\theta = (\alpha,\beta)$. Using the DF mapping $z = d(\theta)$, $\theta = d^{-1}(z)$, it is now fairly straightforward to convert these covariant θ-representations into covariant TF representations.

7.8.1 Covariant Linear Time-Frequency Representations

We first consider linear TF representations (LTFRs). Any LTFR can be written in the form

$$\tilde{L}_x(z) = \langle x, h_z \rangle = \int_t x(t)\, h_z^*(t)\, dt\,,$$

where $h_z(t)$ is a function that depends on the TF point $z = (t,f)$ but not on the signal x.

Definition 7.24 *[25] Let $\{\mathbf{D}_\theta\}_{\theta\in(\mathcal{D},\circ)}$ be a DO on a Hilbert space $\mathcal{X}$, with cocycle c and extended DF e acting on a TF point set $\mathcal{Z}$. An LTFR $\tilde{L}_x(z)$ defined on $\mathcal{Z}$ is called* covariant to the DO $\mathbf{D}_\theta$ *if for all $x \in \mathcal{X}$*

$$\tilde{L}_{\mathbf{D}_{\theta_1}x}(z) \,=\, c\big(d^{-1}(z)\circ\theta_1^{-1},\theta_1\big)\,\tilde{L}_x\big(e(z,\theta_1^{-1})\big)\,, \qquad \text{for all } z\in\mathcal{Z},\ \theta_1\in\mathcal{D}\,, \tag{7.8.68}$$

where $d(\theta)=e(z_0,\theta)$ with some arbitrary but fixed $z_0\in\mathcal{Z}$.

This covariance property states that the LTFR of the displaced signal $(\mathbf{D}_{\theta_1}x)(t)$ equals (possibly up to a phase factor) the TF displaced LTFR of the original signal $x(t)$, $\tilde{L}_x\big(e(z,\theta_1^{-1})\big)$. The covariance property depends on a reference TF point $z_0\in\mathcal{Z}$ that can be chosen arbitrarily; different choices of z_0 result in covariance properties that differ by the phase factor $c\big(d^{-1}(z)\circ\theta_1^{-1},\theta_1\big)$ (since $d(z)$ depends on z_0). Beyond the choice of z_0, there is some arbitrariness regarding the exact definition of the phase factor in (7.8.68). Our specific choice will result in a particularly simple expression for the covariant LTFR $\tilde{L}_x(z)$ (see (7.8.69)).

The next corollary to Theorem 7.13 characterizes all covariant LTFRs.

Corollary 7.25 *[25] Let $\{\mathbf{D}_\theta\}_{\theta\in(\mathcal{D},\circ)}$ be a DO on a Hilbert space $\mathcal{X}$, with extended DF e acting on a TF point set $\mathcal{Z}$. All LTFRs covariant (with reference TF point $z_0\in\mathcal{Z}$) to $\mathbf{D}_\theta$ are given by*

$$\tilde{L}_x(z) \,=\, \big\langle x, \mathbf{D}_{d^{-1}(z)}h\big\rangle \,=\, \int_{t'} x(t')\,\big(\mathbf{D}_{d^{-1}(z)}h\big)^*(t')\,dt', \quad z\in\mathcal{Z}, \tag{7.8.69}$$

where h is an arbitrary function in $\mathcal{X}$ and $d(\theta)=e(z_0,\theta)$.

Proof. Setting $z=d(\theta)$ in the TF covariance property (7.8.68) and using (7.6.46) yields

$$L_{\mathbf{D}_{\theta_1}x}(\theta) \,=\, c(\theta\circ\theta_1^{-1},\theta_1)\,L_x(\theta\circ\theta_1^{-1}), \quad \text{with } L_x(\theta) \triangleq \tilde{L}_x\big(d(\theta)\big)\,. \tag{7.8.70}$$

This is the θ-domain covariance property in (7.5.38). Since the DF mapping d is invertible, this covariance property is strictly equivalent to the original TF covariance property (7.8.68). According to Theorem 7.13, all linear θ-representations satisfying (7.8.70) are given by $L_x(\theta)=\langle x,\mathbf{D}_\theta h\rangle = \int_t x(t)\,(\mathbf{D}_\theta h)^*(t)\,dt$. Since (7.8.68) and (7.8.70) are equivalent via the DF mapping d, all LTFRs satisfying (7.8.68) are then given by

$$\tilde{L}_x(z) \,=\, L_x(\theta)\Big|_{\theta=d^{-1}(z)} \,=\, \big\langle x,\mathbf{D}_{d^{-1}(z)}h\big\rangle\,.$$

□

As an example, consider the operator $\mathbf{S}_{\tau,\nu}=\mathbf{F}_\nu\mathbf{T}_\tau$. This DO has cocycle $c\big((\tau_1,\nu_1),(\tau_2,\nu_2)\big) = e^{-j2\pi\nu_1\tau_2}$ and DF (with $(t_0,f_0)=(0,0)$) $d(\tau,\nu) = (\tau,\nu)$, i.e., $t=\tau$ and $f=\nu$. Thus, the covariance property (7.8.68) becomes

$$\tilde{L}_{\mathbf{S}_{\tau_1,\nu_1}x}(t,f) \,=\, e^{-j2\pi\tau_1(f-\nu_1)}\,\tilde{L}_x(t-\tau_1,f-\nu_1)\,,$$

and the class of all covariant LTFRs is the short-time Fourier transform in (7.1.4), i.e.,

$$\tilde{L}_x(t,f) \;=\; \langle x, \mathbf{S}_{t,f} h\rangle \;=\! \int_{t'} x(t')\, h^*(t'-t)\, e^{-j2\pi f t'} dt'.$$

This is in fact equivalent to the θ-domain covariance property in (7.5.39) and the class of covariant linear θ-representations in (7.5.41), the reason being that the DF is the identity function. Further examples of TF covariance properties and the corresponding classes of covariant LTFRs are provided in Table 7.6—in particular, the TF version of the wavelet transform is obtained for the DO $\mathbf{R}_{\sigma,\tau} = \mathbf{T}_\tau \mathbf{C}_\sigma$—and in Subsection 7.8.3.

7.8.2 Covariant Bilinear Time-Frequency Representations

Next, we consider bilinear TF representations (BTFRs). Any BTFR can be written in the form

$$\tilde{B}_{x,y}(z) \;=\; \langle x, \mathbf{K}_z\, y\rangle \;=\! \int_{t_1}\!\int_{t_2} x(t_1)\, y^*(t_2)\, k_z^*(t_1,t_2)\, dt_1 dt_2\,,$$

where $\mathbf{K}_z$ is a linear operator with kernel $k_z(t_1,t_2)$ that depends on $z = (t,f)$ but not on x, y.

Definition 7.26 *[25, 26, 28] Let $\{\mathbf{D}_\theta\}_{\theta\in(\mathcal{D},\circ)}$ be a DO on a Hilbert space $\mathcal{X}$, with extended DF e acting on a TF point set $\mathcal{Z}$. A BTFR $\tilde{B}_{x,y}(z)$ defined on $\mathcal{Z}$ is called* covariant to the DO $\mathbf{D}_\theta$ *if for all $x,y\in\mathcal{X}$*

$$\tilde{B}_{\mathbf{D}_{\theta_1}x,\mathbf{D}_{\theta_1}y}(z) \;=\; \tilde{B}_{x,y}\big(e(z,\theta_1^{-1})\big)\,, \qquad \textit{for all } z\in\mathcal{Z},\ \theta_1\in\mathcal{D}. \qquad (7.8.71)$$

Apart from the missing cocycle c (and, hence, the independence of a reference TF point z_0), this covariance property is analogous to the corresponding property (7.8.68) in the linear case. The class of all covariant BTFRs is characterized by the following corollary to Theorem 7.15.

Corollary 7.27 *[25] Let $\{\mathbf{D}_\theta\}_{\theta\in(\mathcal{D},\circ)}$ be a DO on a Hilbert space $\mathcal{X}$, with extended DF e acting on a set $\mathcal{Z}$. All BTFRs covariant to $\mathbf{D}_\theta$ are given by*

$$\begin{aligned}\tilde{B}_{x,y}(z) \;&=\; \big\langle x, \mathbf{D}_{d^{-1}(z)} \mathbf{K} \mathbf{D}_{d^{-1}(z)}^{-1}\, y\big\rangle \\ &=\int_{t_1}\!\int_{t_2} x(t_1)\, y^*(t_2)\, \big[\mathbf{D}_{d^{-1}(z)} \mathbf{K} \mathbf{D}_{d^{-1}(z)}^{-1}\big]^*(t_1,t_2)\, dt_1 dt_2\,, \quad z\in\mathcal{Z}, \quad (7.8.72)\end{aligned}$$

where $\mathbf{K}$ is an arbitrary linear operator on $\mathcal{X}$, $\big[\mathbf{D}_\theta \mathbf{K} \mathbf{D}_\theta^{-1}\big](t_1,t_2)$ denotes the kernel of $\mathbf{D}_\theta \mathbf{K} \mathbf{D}_\theta^{-1}$, and $d(\theta) = e(z_0,\theta)$ with arbitrary $z_0\in\mathcal{Z}$.

Proof. The proof is completely analogous to that of Corollary 7.25. In particular, there is

$$\tilde{B}_{x,y}(z) \;=\; B_{x,y}(\theta)\Big|_{\theta=d^{-1}(z)}\,, \qquad (7.8.73)$$

Short-Time Fourier Transform	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = x(t-\alpha)\,, \quad \alpha \in (\mathbb{R},+)$
$\mathbf{B}_\beta$	$(\mathbf{B}_\beta x)(t) = e^{j2\pi\beta t}\, x(t)\,, \quad \beta \in (\mathbb{R},+)$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = e^{-j2\pi\beta_1\alpha_2}\,\mathbf{D}_{\alpha_1+\alpha_2,\,\beta_1+\beta_2}$
displacement function	$d(\alpha,\beta) = (\alpha,\beta)$
covariance property	$\tilde{L}_{\mathbf{D}_{\alpha,\beta}x}(t,f) = e^{-j2\pi\alpha(f-\beta)}\,\tilde{L}_x(t-\alpha, f-\beta)$
covar. lin. TF repres.	$\tilde{L}_x(t,f) = \int_{-\infty}^{\infty} x(t')\, h^*(t'-t)\, e^{-j2\pi f t'}\, dt'$

Wavelet Transform	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+,\cdot)$
$\mathbf{B}_\beta$	$(\mathbf{B}_\beta x)(t) = x(t-\beta)\,, \quad \beta \in (\mathbb{R},+)$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = \mathbf{D}_{\alpha_1\alpha_2,\,\beta_1\alpha_2+\beta_2}$
displacement function	$d(\alpha,\beta) = \big(\beta, \frac{1}{\alpha}\big)$
covariance property	$\tilde{L}_{\mathbf{D}_{\alpha,\beta}x}(t,f) = \tilde{L}_x\big(\frac{t-\beta}{\alpha}, \alpha f\big)\,, \;\; f>0$
covar. lin. TF repres.	$\tilde{L}_x(t,f) = \sqrt{f}\int_{-\infty}^{\infty} x(t')\, h^*\big(f(t'-t)\big)\, dt'\,, \;\; f>0$

Hyperbolic Wavelet Transform	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+,\cdot)$
$\mathbf{B}_\beta$	$(\hat{\mathbf{B}}_\beta X)(f) = e^{-j2\pi\beta\ln f}\, X(f)\,, \quad \beta \in (\mathbb{R},+)\,, \;\; f>0$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = e^{-j2\pi\beta_1\ln\alpha_2}\,\mathbf{D}_{\alpha_1\alpha_2,\,\beta_1+\beta_2}$
displacement function	$d(\alpha,\beta) = \big(\alpha\beta, \frac{1}{\alpha}\big)$
covariance property	$\tilde{L}_{\mathbf{D}_{\alpha,\beta}x}(t,f) = e^{-j2\pi(tf-\beta)\ln\alpha}\,\tilde{L}_x\big(\frac{t-\beta/f}{\alpha}, \alpha f\big)\,, \;\; f>0$
covar. lin. TF repres.	$\tilde{L}_x(t,f) = \frac{1}{\sqrt{f}}\int_0^{\infty} X(f')\, H^*\big(\frac{f'}{f}\big)\, e^{j2\pi t f\ln f'}\, df'\,, \;\; f>0$

Power Wavelet Transform	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+,\cdot)$
$\mathbf{B}_\beta$	$(\hat{\mathbf{B}}_\beta X)(f) = e^{-j2\pi\beta\xi_\kappa(f)}\, X(f)\,, \quad \beta \in (\mathbb{R},+)\,,$ with $\xi_\kappa(f) = \mathrm{sgn}(f)\,\lvert f\rvert^\kappa$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = \mathbf{D}_{\alpha_1\alpha_2,\,\beta_1\xi_\kappa(\alpha_2)+\beta_2}$
displacement function (extended to $f\in\mathbb{R}$)	$d(\alpha,\beta) = \big(\beta\,\xi'_\kappa\big(\frac{1}{\alpha}\big), \pm\frac{1}{\alpha}\big) = \big(\frac{\beta\kappa}{\alpha^{\kappa-1}}, \pm\frac{1}{\alpha}\big)$ (note that $\xi'_\kappa(f) = \kappa\lvert f\rvert^{\kappa-1}$)
covariance property	$\tilde{L}_{\mathbf{D}_{\alpha,\beta}x}(t,f) = \tilde{L}_x\big(\frac{t-\beta\,\xi'_\kappa(f)}{\alpha}, \alpha f\big)$
covar. lin. TF repres. (extended to $f\in\mathbb{R}$)	$\tilde{L}_x(t,f) = \frac{1}{\sqrt{\lvert f\rvert}}\int_{-\infty}^{\infty} X(f')\, H^*\big(\frac{f'}{f}\big)\, e^{j2\pi t\,\xi_\kappa(f')/\xi'_\kappa(f)}\, df'$

TABLE 7.6. *Some covariant linear TF representations. Note that* $\theta = (\alpha,\beta)$, $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$, $X(f)$ *denotes the Fourier transform of* $x(t)$, *and* $\hat{\mathbf{B}}_\beta$ *denotes the frequency-domain version of* $\mathbf{B}_\beta$.

where

$$B_{x,y}(\theta) \;=\; \langle x, \mathbf{D}_\theta \mathbf{K} \mathbf{D}_\theta^{-1} y \rangle \;=\; \int_{t_1}\int_{t_2} x(t_1)\, y^*(t_2) \left[\mathbf{D}_\theta \mathbf{K} \mathbf{D}_\theta^{-1}\right]^*(t_1,t_2)\, dt_1 dt_2$$

is the class of all covariant bilinear θ-representations according to Theorem 7.15. □

Whereas it appears that the class of all covariant BTFRs $\tilde{B}_{x,y}(z)$ in (7.8.72) depends on the reference TF point z_0 via the DF $d(\theta) = e(z_0,\theta)$, this is really not the case. Indeed, if $d(\theta) = e(z_0,\theta)$ and $\tilde{d}(\theta) = e(\tilde{z}_0,\theta)$ are two different DFs of $\mathbf{D}_\theta$ obtained with two different reference TF points z_0 and $\tilde{z}_0$, respectively, then according to (7.6.47) there exists a $\theta' \in \mathcal{D}$ such that $\tilde{d}^{-1}(z) = \theta' \circ d^{-1}(z)$. Hence, we have

$$\begin{aligned}
\mathbf{D}_{\tilde{d}^{-1}(z)} \mathbf{K} \mathbf{D}_{\tilde{d}^{-1}(z)}^{-1} &= \mathbf{D}_{\theta' \circ d^{-1}(z)} \mathbf{K} \mathbf{D}_{\theta' \circ d^{-1}(z)}^{-1} \\
&= c^*\big(\theta', d^{-1}(z)\big)\, \mathbf{D}_{d^{-1}(z)} \mathbf{D}_{\theta'} \mathbf{K} \big[c^*\big(\theta', d^{-1}(z)\big)\, \mathbf{D}_{d^{-1}(z)} \mathbf{D}_{\theta'}\big]^{-1} \\
&= \mathbf{D}_{d^{-1}(z)} \mathbf{D}_{\theta'} \mathbf{K} \mathbf{D}_{\theta'}^{-1} \mathbf{D}_{d^{-1}(z)}^{-1} \;=\; \mathbf{D}_{d^{-1}(z)} \mathbf{K}' \mathbf{D}_{d^{-1}(z)}^{-1} ,
\end{aligned}$$

where $\mathbf{K}' = \mathbf{D}_{\theta'} \mathbf{K} \mathbf{D}_{\theta'}^{-1}$ is again an operator on $\mathcal{X}$. This shows that (7.8.72) with DF $\tilde{d}$ yields the *same* BTFR class as with DF d, albeit with a different parameterization by $\mathbf{K}$.

Examples of covariant BTFRs along with the corresponding covariance property are provided in Table 7.7—in particular, Cohen's class and the affine class of BTFRs are obtained for the DOs $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ and $\mathbf{R}_{\sigma,\tau} = \mathbf{T}_\tau \mathbf{C}_\sigma$, respectively—and in Subsection 7.8.3.

7.8.3 Dual and Affine Modulation and Warping Operators

We shall now specialize our results for the important case where the partial DOs are dual or affine modulation and warping operators. Specifically, the DO is given by (7.5.35), i.e.,

$$\begin{aligned}
(\mathbf{D}_{\alpha,\beta}\, x)(t) \;=\; (\mathbf{M}_\beta \mathbf{W}_\alpha\, x)(t) \;&=\; e^{j2\pi\, \psi_{\mathcal{B}}(\beta)\, \tilde{m}(t)} \sqrt{|w'_\alpha(t)|}\; x\big(w_\alpha(t)\big)\,, \\
&\qquad \alpha \in (\mathcal{A}, \bullet)\,,\;\; \beta \in (\mathcal{B}, *)\,,\;\; t \in \Omega\,,
\end{aligned}$$

with $\tilde{m}(t) = \psi_{\mathcal{A}}(m(t))$. We first consider the case where $\mathbf{M}_\beta$ is *dual* to $\mathbf{W}_\alpha$ (see Subsection 7.3.2), i.e., $m(\Omega) = \mathcal{A}$ and $w_\alpha(t) = m^{-1}\big(m(t) \bullet \alpha^{-1}\big) = \tilde{m}^{-1}\big(\tilde{m}(t) - \psi_{\mathcal{A}}(\alpha)\big)$. From (7.7.65), the inverse DF is obtained as $d^{-1}(t,f) = \Big(m(t)\,,\, \psi_{\mathcal{B}}^{-1}\big(\frac{f}{\tilde{m}'(t)}\big)\Big)$. Evaluating (7.8.69) and (7.8.72), we obtain the covariant LTFRs and BTFRs as

$$\tilde{L}_x(t,f) \;=\; \int_\Omega x(t')\, h^*\big(w_{m(t)}(t')\big) \sqrt{\big|w'_{m(t)}(t')\big|}\; e^{-j2\pi f\, \tilde{m}(t')/\tilde{m}'(t)}\, dt'$$

Cohen's Class	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = x(t-\alpha)\,, \quad \alpha \in (\mathbb{R},+)$
$\mathbf{B}_\beta$	$(\mathbf{B}_\beta\, x)(t) = e^{j2\pi\beta t}\, x(t)\,, \quad \beta \in (\mathbb{R},+)$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = e^{-j2\pi\beta_1\alpha_2}\,\mathbf{D}_{\alpha_1+\alpha_2,\,\beta_1+\beta_2}$
displacement function	$d(\alpha,\beta) = (\alpha,\beta)$
covariance property	$\tilde{B}_{\mathbf{D}_{\alpha,\beta}x,\mathbf{D}_{\alpha,\beta}y}(t,f) = \tilde{B}_{x,y}(t-\alpha, f-\beta)$
covar. bilin. TF repres.	$\tilde{B}_{x,y}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, y^*(t_2)\, k^*(t_1-t, t_2-t) \cdot e^{-j2\pi f(t_1-t_2)}\, dt_1\, dt_2$

Affine Class	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+,\cdot)$
$\mathbf{B}_\beta$	$(\mathbf{B}_\beta\, x)(t) = x(t-\beta)\,, \quad \beta \in (\mathbb{R},+)$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = \mathbf{D}_{\alpha_1\alpha_2,\,\beta_1\alpha_2+\beta_2}$
displacement function	$d(\alpha,\beta) = \big(\beta, \frac{1}{\alpha}\big)$
covariance property	$\tilde{B}_{\mathbf{D}_{\alpha,\beta}x,\mathbf{D}_{\alpha,\beta}y}(t,f) = \tilde{B}_{x,y}\big(\frac{t-\beta}{\alpha}, \alpha f\big)\,, \ f>0$
covar. bilin. TF repres.	$\tilde{B}_{x,y}(t,f) = f\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\, y^*(t_2) \cdot k^*\big(f(t_1-t), f(t_2-t)\big)\, dt_1 dt_2\,, \ f>0$

Hyperbolic Class	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+,\cdot)$
$\mathbf{B}_\beta$	$(\hat{\mathbf{B}}_\beta X)(f) = e^{-j2\pi\beta \ln f}\, X(f)\,, \quad \beta \in (\mathbb{R},+)\,, \ f>0$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = e^{-j2\pi\beta_1 \ln\alpha_2}\,\mathbf{D}_{\alpha_1\alpha_2,\,\beta_1+\beta_2}$
displacement function	$d(\alpha,\beta) = \big(\alpha\beta, \frac{1}{\alpha}\big)$
covariance property	$\tilde{B}_{\mathbf{D}_{\alpha,\beta}x,\mathbf{D}_{\alpha,\beta}y}(t,f) = \tilde{B}_{x,y}\big(\frac{t-\beta/f}{\alpha}, \alpha f\big)\,, \ f>0$
covar. bilin. TF repres.	$\tilde{B}_{x,y}(t,f) = \frac{1}{f}\int_0^{\infty}\int_0^{\infty} X(f_1)\, Y^*(f_2)\, \hat{k}^*\big(\frac{f_1}{f}, \frac{f_2}{f}\big) \cdot e^{j2\pi tf \ln(f_1/f_2)}\, df_1 df_2\,, \ f>0$

Power Classes	
$\mathbf{A}_\alpha$	$(\mathbf{A}_\alpha x)(t) = \frac{1}{\sqrt{\alpha}}\, x\big(\frac{t}{\alpha}\big)\,, \quad \alpha \in (\mathbb{R}^+,\cdot)$
$\mathbf{B}_\beta$	$(\hat{\mathbf{B}}_\beta X)(f) = e^{-j2\pi\beta\xi_\kappa(f)}\, X(f)\,, \quad \beta \in (\mathbb{R},+)\,,$ with $\xi_\kappa(f) = \mathrm{sgn}(f)\,|f|^\kappa$
composition property	$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} = \mathbf{D}_{\alpha_1\alpha_2,\,\beta_1\xi_\kappa(\alpha_2)+\beta_2}$
displacement function (extended to $f\in\mathbb{R}$)	$d(\alpha,\beta) = \big(\beta\,\xi_\kappa'\big(\frac{1}{\alpha}\big), \pm\frac{1}{\alpha}\big) = \big(\frac{\beta\kappa}{\alpha^{\kappa-1}}, \pm\frac{1}{\alpha}\big)$ (note that $\xi_\kappa'(f) = \kappa\,|f|^{\kappa-1}$)
covariance property	$\tilde{B}_{\mathbf{D}_{\alpha,\beta}x,\mathbf{D}_{\alpha,\beta}y}(t,f) = \tilde{B}_{x,y}\big(\frac{t-\beta\,\xi_\kappa'(f)}{\alpha}, \alpha f\big)$
covar. bilin. TF repres. (extended to $f\in\mathbb{R}$)	$\tilde{B}_{x,y}(t,f) = \frac{1}{\lvert f\rvert}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f_1)\, Y^*(f_2)\, \hat{k}^*\big(\frac{f_1}{f}, \frac{f_2}{f}\big) \cdot e^{j2\pi t[\xi_\kappa(f_1)-\xi_\kappa(f_2)]/\xi_\kappa'(f)}\, df_1 df_2$

TABLE 7.7. *Some covariant bilinear/quadratic TF representations. Note that* $\theta = (\alpha,\beta)$, $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$, $X(f)$ *denotes the Fourier transform of* $x(t)$, *and* $\hat{\mathbf{B}}_\beta$ *denotes the frequency-domain version of* $\mathbf{B}_\beta$.

$$\tilde{B}_{x,y}(t,f) = \int_\Omega \int_\Omega x(t_1)\, y^*(t_2)\, k^*\big(w_{m(t)}(t_1), w_{m(t)}(t_2)\big) \cdot \sqrt{\left|w'_{m(t)}(t_1)\, w'_{m(t)}(t_2)\right|}\, e^{-j2\pi f\, [\tilde{m}(t_1)-\tilde{m}(t_2)]/\tilde{m}'(t)}\, dt_1 dt_2\,,$$

for $(t,f) \in \Omega \times \mathbb{R}$. These TF representations satisfy the following covariance properties (see (7.8.68), (7.8.71), and (7.7.64)),

$$\tilde{L}_{\mathbf{D}_{\alpha,\beta}x}(t,f) = \exp\!\left(-j2\pi\, \psi_{\mathcal{A}}(\alpha) \left[\frac{f}{\tilde{m}'(t)} - \psi_{\mathcal{B}}(\beta)\right]\right) \cdot \tilde{L}_x\!\left(w_\alpha(t)\,,\, \frac{f - \psi_{\mathcal{B}}(\beta)\, \tilde{m}'(t)}{w'_\alpha(t)}\right)$$

$$\tilde{B}_{\mathbf{D}_{\alpha,\beta}x,\mathbf{D}_{\alpha,\beta}y}(t,f) = \tilde{B}_{x,y}\!\left(w_\alpha(t)\,,\, \frac{f - \psi_{\mathcal{B}}(\beta)\, \tilde{m}'(t)}{w'_\alpha(t)}\right).$$

Next, let $\mathbf{M}_\beta$ be *affine* to $\mathbf{W}_\alpha$ (see Subsection 7.4.2), i.e., $\tilde{m}(\Omega) = \mathbb{R}^+$ or $\tilde{m}(\Omega) = \mathbb{R}^-$ and $w_\alpha(t) = \tilde{m}^{-1}\big(\tilde{m}(t) \exp(\psi_{\mathcal{A}}(\alpha))\big)$. From (7.7.67), the inverse DF follows as $d^{-1}(t,f) = \Big(\psi_{\mathcal{A}}^{-1}\big(-\ln|\tilde{m}(t)|\big)\,,\, \psi_{\mathcal{B}}^{-1}\big(\frac{f}{\tilde{m}'(t)}\big)\Big)$. Evaluating (7.8.69) and (7.8.72), we obtain the covariant LTFRs and BTFRs as

$$\tilde{L}_x(t,f) = \int_\Omega x(t')\, h^*\big(w_{\chi(t)}(t')\big) \sqrt{\left|w'_{\chi(t)}(t')\right|}\; e^{-j2\pi f\, \tilde{m}(t')/\tilde{m}'(t)}\, dt'$$

$$\tilde{B}_{x,y}(t,f) = \int_\Omega \int_\Omega x(t_1)\, y^*(t_2)\, k^*\big(w_{\chi(t)}(t_1), w_{\chi(t)}(t_2)\big) \cdot \sqrt{\left|w'_{\chi(t)}(t_1)\, w'_{\chi(t)}(t_2)\right|}\, e^{-j2\pi f\, [\tilde{m}(t_1)-\tilde{m}(t_2)]/\tilde{m}'(t)}\, dt_1 dt_2\,,$$

where $\chi(t) \triangleq \psi_{\mathcal{A}}^{-1}\big(-\ln|\tilde{m}(t)|\big)$. These TF representations satisfy the covariance properties

$$\tilde{L}_{\mathbf{D}_{\alpha,\beta}x}(t,f) = \tilde{L}_x\!\left(w_\alpha(t)\,,\, \frac{f - \psi_{\mathcal{B}}(\beta)\, \tilde{m}'(t)}{w'_\alpha(t)}\right)$$

$$\tilde{B}_{\mathbf{D}_{\alpha,\beta}x,\mathbf{D}_{\alpha,\beta}y}(t,f) = \tilde{B}_{x,y}\!\left(w_\alpha(t)\,,\, \frac{f - \psi_{\mathcal{B}}(\beta)\, \tilde{m}'(t)}{w'_\alpha(t)}\right).$$

7.9 The Characteristic Function Method

Besides covariance properties, which are the main subject of this chapter, *marginal properties* are of importance in the context of BTFRs. Marginal properties state that integration of a BTFR along certain curves in the TF plane yields corresponding energy densities of the signals, thus suggesting the BTFR's interpretation as a joint TF energy distribution. The shape of the curves and the definition of the energy densities depend on a pair of unitary operator families (unitary group representations) $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$.

The *characteristic function method* [2, 34, 57–60] aims at constructing joint "(a,b)-distributions," i.e., signal representations that are functions of real variables a, b and satisfy marginal properties in the (a,b)-domain. The nature of the variables a, b depends on the operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$. Using a mapping between the (a,b)-plane and the TF plane (the *localization function*, see Subsection 7.9.2), the (a,b)-distributions can then be converted into BTFRs that satisfy marginal properties in the TF domain.

If we are given two unitary operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ for which $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$ is a DO, then we can apply both the covariance method and the characteristic function method to construct classes of BTFRs. In general, the classes obtained will be different. However, in Subsection 7.9.3 we will show that in the special case of dual operators, the covariance method and characteristic function method yield equivalent results.

7.9.1 (a,b) Energy Distributions

Let $\{\mathbf{A}_\alpha\}_{\alpha\in(\mathcal{A},\bullet)}$ and $\{\mathbf{B}_\beta\}_{\beta\in(\mathcal{B},*)}$ be unitary representations of groups $(\mathcal{A},\bullet)$ and $(\mathcal{B},*)$ that are isomorphic to $(\mathbb{R},+)$ by $\psi_\mathcal{A}$ and $\psi_\mathcal{B}$, respectively. Let $\{u_b^{\mathbf{A}}(t)\}_{b\in(\tilde{\mathcal{A}},\tilde{\bullet})}$ and $\{u_a^{\mathbf{B}}(t)\}_{a\in(\tilde{\mathcal{B}},\tilde{*})}$ denote the (generalized) eigenfunctions of $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$, respectively, where $(\tilde{\mathcal{A}},\tilde{\bullet})$ and $(\tilde{\mathcal{B}},\tilde{*})$ are again isomorphic to $(\mathbb{R},+)$ by $\psi_{\tilde{\mathcal{A}}}$ and $\psi_{\tilde{\mathcal{B}}}$, respectively. According to the characteristic function method, (a,b)-distributions (with $a\in\tilde{\mathcal{B}}$ and $b\in\tilde{\mathcal{A}}$ the parameters of the eigenfunctions $u_a^{\mathbf{B}}(t)$ and $u_b^{\mathbf{A}}(t)$, respectively) are constructed as

$$\begin{aligned} P_{x,y}(a,b) &= \int_\mathcal{A}\int_\mathcal{B} Q_{x,y}(\alpha,\beta)\, e^{-j2\pi[\psi_\mathcal{A}(\alpha)\psi_{\tilde{\mathcal{A}}}(b)\,-\,\psi_\mathcal{B}(\beta)\psi_{\tilde{\mathcal{B}}}(a)]}\, d\psi_\mathcal{A}(\alpha)\, d\psi_\mathcal{B}(\beta) \qquad (7.9.74)\\ &= \int_\mathbb{R}\int_\mathbb{R} Q_{x,y}\big(\psi_\mathcal{A}^{-1}(\alpha'),\psi_\mathcal{B}^{-1}(\beta')\big)\, e^{-j2\pi[\alpha'\psi_{\tilde{\mathcal{A}}}(b)\,-\,\beta'\psi_{\tilde{\mathcal{B}}}(a)]}\, d\alpha'\, d\beta'\,, \end{aligned}$$

with the "characteristic function" $Q_{x,y}(\alpha,\beta)$ defined as

$$Q_{x,y}(\alpha,\beta) \triangleq \phi(\alpha,\beta)\,\langle x, \mathbf{Q}_{\alpha,\beta}\, y\rangle = \phi(\alpha,\beta)\int_t x(t)\big(\mathbf{Q}_{\alpha,\beta}\, y\big)^*(t)\, dt\,. \qquad (7.9.75)$$

Here, $\phi : \mathcal{A}\times\mathcal{B}\to\mathbb{C}$ is an arbitrary "kernel" function and $\mathbf{Q}_{\alpha,\beta}$ is any operator satisfying $\mathbf{Q}_{\alpha_0,\beta} = \mathbf{B}_\beta$ and $\mathbf{Q}_{\alpha,\beta_0} = \mathbf{A}_\alpha$. Examples of such "characteristic function operators" are $\mathbf{Q}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$, $\mathbf{Q}_{\alpha,\beta} = \mathbf{A}_\alpha\mathbf{B}_\beta$, $\mathbf{Q}_{\alpha,\beta} = \frac{1}{2}(\mathbf{A}_\alpha\mathbf{B}_\beta + \mathbf{B}_\beta\mathbf{A}_\alpha)$, $\mathbf{Q}_{\alpha,\beta} = \mathbf{A}_{\alpha^{1/2}}\mathbf{B}_\beta\mathbf{A}_{\alpha^{1/2}}$ (where $\alpha^{1/2}$ is defined by $\psi_\mathcal{A}(\alpha^{1/2}) = \psi_\mathcal{A}(\alpha)/2$), etc. Assuming that $\phi(\alpha,\beta)$ and $\mathbf{Q}_{\alpha,\beta}$ do not depend on the signals $x(t)$ and $y(t)$, $P_{x,y}(a,b)$ is a *bilinear* signal representation. If $\phi(\alpha,\beta)$ satisfies $\phi(\alpha_0,\beta) = \phi(\alpha,\beta_0) = 1$, then it can be shown [34] that (assuming suitable parameterization of the eigenfunctions) $P_{x,x}(a,b)$

satisfies the marginal properties

$$\int_{\tilde{\mathcal{A}}} P_{x,x}(a,b)\, d\psi_{\tilde{\mathcal{A}}}(b) \;=\; |\langle x, u_a^{\mathbf{B}}\rangle|^2, \qquad \int_{\tilde{\mathcal{B}}} P_{x,x}(a,b)\, d\psi_{\tilde{\mathcal{B}}}(a) \;=\; |\langle x, u_b^{\mathbf{A}}\rangle|^2. \tag{7.9.76}$$

7.9.2 Time-Frequency Energy Distributions

Since we are primarily interested in TF distributions, we will now convert the (a,b)-distributions $P_{x,y}(a,b)$ constructed above into TF distributions. This conversion uses an invertible mapping $(a,b) \to (t,f)$ that will be termed the *localization function* (LF) [10, 26, 28, 30]. To construct the LF, we consider the eigenfunctions $\{u_b^{\mathbf{A}}(t)\}_{b\in(\tilde{\mathcal{A}},\tilde{\bullet})}$ and $\{u_a^{\mathbf{B}}(t)\}_{a\in(\tilde{\mathcal{B}},\tilde{*})}$ of $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$, respectively. The TF locus of $u_b^{\mathbf{A}}(t)$ is characterized by the instantaneous frequency $\nu\{u_b^{\mathbf{A}}\}(t)$ or the group delay $\tau\{u_b^{\mathbf{A}}\}(f)$, whichever exists. Here, e.g., we assume existence of the instantaneous frequency $\nu\{u_b^{\mathbf{A}}\}(t)$. Similarly, we assume existence of the group delay $\tau\{u_a^{\mathbf{B}}\}(f)$ of $u_a^{\mathbf{B}}(t)$. The TF loci of $u_b^{\mathbf{A}}(t)$ and $u_a^{\mathbf{B}}(t)$ being described by $\nu\{u_b^{\mathbf{A}}\}(t)$ and $\tau\{u_a^{\mathbf{B}}\}(f)$, respectively, the TF point (t,f) that corresponds to the eigenfunction parameter pair (a,b) is defined by the intersection of these two TF curves, i.e., (t,f) is given by the solution to the system of equations

$$\nu\{u_b^{\mathbf{A}}\}(t) = f\,, \qquad \tau\{u_a^{\mathbf{B}}\}(f) = t\,. \tag{7.9.77}$$

If this system of equations has a unique solution (t,f) for each $(a,b) \in \tilde{\mathcal{B}} \times \tilde{\mathcal{A}}$, then we can write $(t,f) = l(a,b)$ with a function $l : \tilde{\mathcal{B}} \times \tilde{\mathcal{A}} \to \mathcal{Z}$ that will be termed the LF of the operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$. We moreover assume that, conversely, the system of equations (7.9.77) has a unique solution $(a,b) \in \tilde{\mathcal{B}} \times \tilde{\mathcal{A}}$ for each $(t,f) \in \mathcal{Z}$; the LF then is invertible and we can write $(a,b) = l^{-1}(t,f)$ for all $(t,f) \in \mathcal{Z}$.

Using the LF mapping, the (a,b)-distribution $P_{x,y}(a,b)$ in (7.9.74) can now be converted into a TF distribution $\tilde{P}_{x,y}(t,f)$ according to

$$\tilde{P}_{x,y}(t,f) \;\triangleq\; P_{x,y}(a,b)\Big|_{(a,b)\,=\,l^{-1}(t,f)}\,. \tag{7.9.78}$$

If the kernel $\phi(\alpha,\beta)$ in (7.9.75) satisfies $\phi(\alpha_0,\beta) = \phi(\alpha,\beta_0) = 1$, then $\tilde{P}_{x,x}(t,f)$ satisfies the following marginal properties (cf. (7.9.76)),

$$\int_{\tilde{\mathcal{A}}} \tilde{P}_{x,x}\big(l(a,b)\big)\, d\psi_{\tilde{\mathcal{A}}}(b) = |\langle x, u_a^{\mathbf{B}}\rangle|^2, \qquad \int_{\tilde{\mathcal{B}}} \tilde{P}_{x,x}\big(l(a,b)\big)\, d\psi_{\tilde{\mathcal{B}}}(a) = |\langle x, u_b^{\mathbf{A}}\rangle|^2. \tag{7.9.79}$$

As an example illustrating the construction of the LF, let us consider the case where $\mathbf{A}_\alpha = \mathbf{W}_\alpha$ is a warping operator with warping function $w_\alpha(t) = n^{-1}\big(n(t) \bullet \alpha^{-1}\big)$, where $n : \Omega \to \mathcal{A}$, and $\mathbf{B}_\beta = \mathbf{M}_\beta$ is a modulation operator with modulation function $m : \Omega \to \tilde{\mathcal{B}}$. According to Subsections 7.2.3, 7.2.4, and 7.7.2, the eigenfunctions of $\mathbf{W}_\alpha$ are $u_b^{\mathbf{W}}(t) = \sqrt{|\tilde{n}'(t)|}\, e^{j2\pi\psi_{\tilde{\mathcal{A}}}(b)\,\tilde{n}(t)}$, with $b \in \tilde{\mathcal{A}}$ and $\tilde{n}(t) = \psi_{\mathcal{A}}(n(t))$, and the eigenfunctions of $\mathbf{M}_\beta$ are $u_a^{\mathbf{M}}(t) = \sqrt{|\tilde{m}'(t)|}\,\delta\big(\tilde{m}(t) - \psi_{\tilde{\mathcal{B}}}(a)\big)$ with $a \in \tilde{\mathcal{B}}$ and $\tilde{m}(t) = \psi_{\tilde{\mathcal{B}}}(m(t))$. We assume $m(\Omega) = \tilde{\mathcal{B}}$ and $|\tilde{n}'(t)| \neq 0$. With $\nu\{u_b^{\mathbf{W}}\}(t) = \psi_{\tilde{\mathcal{A}}}(b)\,\tilde{n}'(t)$ being the instantaneous frequency of $u_b^{\mathbf{W}}(t)$ and $\tau\{u_a^{\mathbf{M}}\}(f) \equiv m^{-1}(a)$ being the group delay of $u_a^{\mathbf{M}}(t)$, solution of (7.9.77) yields the LF as

$$l(a,b) \;=\; \big(m^{-1}(a), \psi_{\tilde{\mathcal{A}}}(b)\,\tilde{n}'\big(m^{-1}(a)\big)\big)\,. \tag{7.9.80}$$

Inserting for $l(a,b)$, $u_a^{\mathbf{M}}(t)$, and $u_b^{\mathbf{W}}(t)$, the TF-domain marginal properties in (7.9.79) are obtained as

$$\int_{\mathbb{R}} \tilde{P}_{x,x}(t,f)\,df \;=\; \left|\frac{\tilde{n}'(t)}{\tilde{m}'(t)}\right| |x(t)|^2, \qquad t \in \Omega$$

$$\int_{\Omega} \tilde{P}_{x,x}\big(t, c\tilde{n}'(t)\big)\,|\tilde{m}'(t)|\,dt \;=\; \left|\int_{\Omega} x(t)\,e^{-j2\pi c\,\tilde{n}(t)}\sqrt{|\tilde{n}'(t)|}\,dt\right|^2, \qquad c \in \mathbb{R}.$$

In particular, if $\mathbf{M}_\beta$ is dual to $\mathbf{W}_\alpha$, then $\tilde{n}(t) = \tilde{m}(t)$ and the marginal properties simplify as

$$\int_{\mathbb{R}} \tilde{P}_{x,x}(t,f)\,df \;=\; |x(t)|^2, \qquad t \in \Omega$$

$$\int_{\Omega} \tilde{P}_{x,x}\big(t, c\tilde{m}'(t)\big)\,|\tilde{m}'(t)|\,dt \;=\; \left|\int_{\Omega} x(t)\,e^{-j2\pi c\,\tilde{m}(t)}\sqrt{|\tilde{m}'(t)|}\,dt\right|^2, \qquad c \in \mathbb{R}.$$

7.9.3 Equivalence Results for Dual Operators

As before, let the groups $(\mathcal{A}, \bullet)$ and $(\mathcal{B}, *)$ be isomorphic to $(\mathbb{R}, +)$ by $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{B}}$, respectively. We now consider the case where $\{\mathbf{B}_\beta\}_{\beta\in(\mathcal{B},*)}$ is *dual* to $\{\mathbf{A}_\alpha\}_{\alpha\in(\mathcal{A},\bullet)}$, i.e. (see (7.3.23)),

$$\mathbf{B}_\beta\mathbf{A}_\alpha \;=\; e^{j2\pi\,\psi_{\mathcal{A}}(\alpha)\,\psi_{\mathcal{B}}(\beta)}\,\mathbf{A}_\alpha\mathbf{B}_\beta\,. \tag{7.9.81}$$

Furthermore, according to Theorem 7.6, $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$ is a DO satisfying

$$\mathbf{D}_{\alpha_2,\beta_2}\mathbf{D}_{\alpha_1,\beta_1} \;=\; e^{-j2\pi\,\psi_{\mathcal{A}}(\alpha_2)\,\psi_{\mathcal{B}}(\beta_1)}\mathbf{D}_{\alpha_1\bullet\alpha_2,\,\beta_1*\beta_2}\,.$$

Here, the displacement parameter $\theta = (\alpha,\beta)$ belongs to the commutative group $(\mathcal{A}\times\mathcal{B}, \circ)$ with group operation $(\alpha_1,\beta_1)\circ(\alpha_2,\beta_2) = (\alpha_1\bullet\alpha_2, \beta_1*\beta_2)$; this group is isomorphic to $(\mathbb{R}^2, +)$ by $\psi(\alpha,\beta) = \big(\psi_{\mathcal{A}}(\alpha), \psi_{\mathcal{B}}(\beta)\big)$.

Using the covariance method (see Subsection 7.5.3), we can construct the class $\big\{B_{x,y}(\alpha,\beta)\big\}$ of all bilinear (α,β)-representations $B_{x,y}(\alpha,\beta)$ that are covariant to the DO $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$; this class is parameterized by an operator $\mathbf{K}$ or, equivalently, its kernel $k(t_1,t_2)$. On the other hand, using the characteristic function method we can construct the class $\big\{P_{x,y}(a,b)\big\}$ of all bilinear (a,b)-distributions $P_{x,y}(a,b)$ that satisfy marginal properties related to the operators $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$; this class is parameterized by a kernel function $\phi(\alpha,\beta)$. (Here, the specific definition of the operator $\mathbf{Q}_{\alpha,\beta}$ in (7.9.75) is immaterial: due to the commutation relation (7.9.81), different operators $\mathbf{Q}_{\alpha,\beta}$ are equal up to complex factors that can be incorporated in the kernel $\phi(\alpha,\beta)$ and thus lead to the same class of distributions $\{P_{x,y}(a,b)\}$ [59].) The following theorem states that the classes $\{B_{x,y}(\alpha,\beta)\}$ and $\{P_{x,y}(a,b)\}$ are equivalent in the dual case considered.

Theorem 7.28 *[28] Let* $\{\mathbf{B}_\beta\}_{\beta\in(\mathcal{B},*)}$ *be dual to* $\{\mathbf{A}_\alpha\}_{\alpha\in(\mathcal{A},\bullet)}$. *Let* $\{P_{x,y}(a,b)\}$ *be the class of (a,b)-distributions based on $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$, as given by (7.9.74), (7.9.75) with $\mathbf{Q}_{\alpha,\beta}=\mathbf{B}_\beta\mathbf{A}_\alpha$ (in the dual case, this choice implies no loss of generality) and eigenfunction parameter groups $(\tilde{\mathcal{A}},\tilde{\bullet})=(\mathcal{B},*)$, $(\tilde{\mathcal{B}},\tilde{*})=(\mathcal{A},\bullet)$ (hence $a\in(\mathcal{A},\bullet)$, $b\in(\mathcal{B},*)$). Let $\{B_{x,y}(\alpha,\beta)\}$ be the class of covariant (α,β)-representations based on the DO $\mathbf{D}_{\alpha,\beta}=\mathbf{B}_\beta\mathbf{A}_\alpha$, as given by (7.5.44). Then these classes are identical, i.e.,*

$$\{P_{x,y}(a,b)\} = \{B_{x,y}(a,b)\}\,. \tag{7.9.82}$$

Moreover, specific representations of either class are identical, i.e.,

$$P_{x,y}(a,b) \;\equiv\; B_{x,y}(a,b)\,, \tag{7.9.83}$$

if the operator $\mathbf{K}$ defining $B_{x,y}(\alpha,\beta)$ is related to the kernel $\phi(\alpha,\beta)$ defining $P_{x,y}(a,b)$ as

$$\mathbf{K} \;=\; \int_{\mathcal{A}}\int_{\mathcal{B}} \phi^*(\alpha,\beta)\,\mathbf{D}_{\alpha,\beta}\,d\psi_{\mathcal{A}}(\alpha)\,d\psi_{\mathcal{B}}(\beta)\,. \tag{7.9.84}$$

Proof. With $\mathbf{Q}_{\alpha,\beta}=\mathbf{B}_\beta\mathbf{A}_\alpha=\mathbf{D}_{\alpha,\beta}$ and $(\tilde{\mathcal{A}},\tilde{\bullet})=(\mathcal{B},*)$, $(\tilde{\mathcal{B}},\tilde{*})=(\mathcal{A},\bullet)$, the (a,b)-distributions obtained by the characteristic function method (see (7.9.74), (7.9.75)) are given by

$$\begin{aligned} P_{x,y}(a,b) \;&=\; \int_{\mathcal{A}}\int_{\mathcal{B}} \phi(\alpha,\beta)\,\langle x,\mathbf{D}_{\alpha,\beta}\,y\rangle\, e^{-j2\pi[\psi_{\mathcal{A}}(\alpha)\psi_{\mathcal{B}}(b)\,-\,\psi_{\mathcal{B}}(\beta)\psi_{\mathcal{A}}(a)]} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\cdot\; d\psi_{\mathcal{A}}(\alpha)\,d\psi_{\mathcal{B}}(\beta) \\ &=\; \Big\langle x, \Big[\int_{\mathcal{A}}\int_{\mathcal{B}} \phi^*(\alpha,\beta)\,\mathbf{D}_{\alpha,\beta}\, e^{j2\pi[\psi_{\mathcal{A}}(\alpha)\psi_{\mathcal{B}}(b)\,-\,\psi_{\mathcal{B}}(\beta)\psi_{\mathcal{A}}(a)]} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\cdot\; d\psi_{\mathcal{A}}(\alpha)\,d\psi_{\mathcal{B}}(\beta)\Big]\,y\Big\rangle \\ &=\; \big\langle x, \mathbf{K}_{a,b}\,y\big\rangle\,, \end{aligned}$$

with

$$\mathbf{K}_{a,b} \;=\; \int_{\mathcal{A}}\int_{\mathcal{B}} \phi^*(\alpha,\beta)\,\mathbf{D}_{\alpha,\beta}\, e^{j2\pi[\psi_{\mathcal{A}}(\alpha)\psi_{\mathcal{B}}(b)\,-\,\psi_{\mathcal{B}}(\beta)\psi_{\mathcal{A}}(a)]}\,d\psi_{\mathcal{A}}(\alpha)\,d\psi_{\mathcal{B}}(\beta)\,.$$

On the other hand, the (α,β)-representations covariant to $\mathbf{D}_{\alpha,\beta}=\mathbf{B}_\beta\mathbf{A}_\alpha$ are given by (see (7.5.44))

$$B_{x,y}(\alpha,\beta) \;=\; \big\langle x, \mathbf{D}_{\alpha,\beta}\mathbf{K}\mathbf{D}_{\alpha,\beta}^{-1}\,y\big\rangle\,.$$

Hence, there is $P_{x,y}(a,b)=B_{x,y}(a,b)$ if and only if $\mathbf{K}_{a,b}$ is of the form $\mathbf{D}_{a,b}\mathbf{K}\mathbf{D}_{a,b}^{-1}$, i.e., if and only if

$$\mathbf{D}_{a,b}\mathbf{K}\mathbf{D}_{a,b}^{-1} = \int_{\mathcal{A}}\int_{\mathcal{B}} \phi^*(\alpha,\beta)\,\mathbf{D}_{\alpha,\beta}\, e^{j2\pi[\psi_{\mathcal{A}}(\alpha)\psi_{\mathcal{B}}(b) - \psi_{\mathcal{B}}(\beta)\psi_{\mathcal{A}}(a)]} \cdot d\psi_{\mathcal{A}}(\alpha)\, d\psi_{\mathcal{B}}(\beta)$$

or, equivalently,

$$\mathbf{K} = \int_{\mathcal{A}}\int_{\mathcal{B}} \phi^*(\alpha,\beta)\,\mathbf{D}_{a,b}^{-1}\mathbf{D}_{\alpha,\beta}\mathbf{D}_{a,b}\, e^{j2\pi[\psi_{\mathcal{A}}(\alpha)\psi_{\mathcal{B}}(b) - \psi_{\mathcal{B}}(\beta)\psi_{\mathcal{A}}(a)]} \cdot d\psi_{\mathcal{A}}(\alpha)\, d\psi_{\mathcal{B}}(\beta)\,. \quad (7.9.85)$$

With (7.2.13) and (7.3.24), there is (note that according to (7.3.24), the cocycle of $\mathbf{D}_{\alpha,\beta}$ is $c\big((\alpha_1,\beta_1),(\alpha_2,\beta_2)\big) = e^{-j2\pi\,\psi_{\mathcal{A}}(\alpha_2)\,\psi_{\mathcal{B}}(\beta_1)}$)

$$\begin{aligned}
\mathbf{D}_{a,b}^{-1}\mathbf{D}_{\alpha,\beta}\mathbf{D}_{a,b} &= e^{j2\pi\psi_{\mathcal{A}}(a^{-1})\psi_{\mathcal{B}}(b)}\,\mathbf{D}_{(a,b)^{-1}}\mathbf{D}_{\alpha,\beta}\mathbf{D}_{a,b} \\
&= e^{-j2\pi\psi_{\mathcal{A}}(a)\psi_{\mathcal{B}}(b)}\,\mathbf{D}_{a^{-1},b^{-1}}\, e^{-j2\pi\psi_{\mathcal{A}}(\alpha)\psi_{\mathcal{B}}(b)}\,\mathbf{D}_{a\bullet\alpha,\,b*\beta} \\
&= e^{-j2\pi\psi_{\mathcal{A}}(a)\psi_{\mathcal{B}}(b)}\, e^{-j2\pi\psi_{\mathcal{A}}(\alpha)\psi_{\mathcal{B}}(b)}\, e^{-j2\pi\psi_{\mathcal{A}}(a^{-1})\psi_{\mathcal{B}}(b*\beta)} \\
&\qquad \cdot \mathbf{D}_{a\bullet\alpha\bullet a^{-1},\,b*\beta*b^{-1}} \\
&= e^{-j2\pi\psi_{\mathcal{A}}(\alpha)\psi_{\mathcal{B}}(b)}\, e^{j2\pi\psi_{\mathcal{A}}(a)\psi_{\mathcal{B}}(\beta)}\,\mathbf{D}_{\alpha,\beta}\,,
\end{aligned}$$

where in the last step we used that $\bullet$ and $*$ are commutative. Hence, (7.9.85) becomes (7.9.84):

$$\begin{aligned}
\mathbf{K} &= \int_{\mathcal{A}}\int_{\mathcal{B}} \phi^*(\alpha,\beta)\, e^{-j2\pi\psi_{\mathcal{A}}(\alpha)\psi_{\mathcal{B}}(b)}\, e^{j2\pi\psi_{\mathcal{A}}(a)\psi_{\mathcal{B}}(\beta)} \\
&\qquad \cdot \mathbf{D}_{\alpha,\beta}\, e^{j2\pi[\psi_{\mathcal{A}}(\alpha)\psi_{\mathcal{B}}(b) - \psi_{\mathcal{B}}(\beta)\psi_{\mathcal{A}}(a)]}\, d\psi_{\mathcal{A}}(\alpha)\, d\psi_{\mathcal{B}}(\beta) \\
&= \int_{\mathcal{A}}\int_{\mathcal{B}} \phi^*(\alpha,\beta)\,\mathbf{D}_{\alpha,\beta}\, d\psi_{\mathcal{A}}(\alpha)\, d\psi_{\mathcal{B}}(\beta)\,.
\end{aligned}$$

This shows that $P_{x,y}(a,b) = B_{x,y}(a,b)$ if and only if $\phi(\alpha,\beta)$ and $\mathbf{K}$ are related according to (7.9.84).

Finally, it is shown in [45] (see also [56]) that every $\mathbf{K}$ can be represented according to (7.9.84) with some $\phi(\alpha,\beta)$. We hence conclude that the entire classes $\{P_{x,y}(a,b)\}$ and $\{B_{x,y}(a,b)\}$ are equivalent. □

The equivalence of characteristic function method and covariance method in the dual case applies not only to the (a,b)- and (α,β)-representations but carries over to the BTFRs derived from them through the LF mapping and the DF mapping, respectively. This is due to the fact that *in the dual case, the LF and DF are themselves equivalent.* To show this, we recall from (7.7.60) that the extended DF of $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$ is obtained by composing the extended DF $e_{\mathbf{A}}$ of $\mathbf{A}_\alpha$ with the extended DF $e_{\mathbf{B}}$ of $\mathbf{B}_\beta$. This composition is illustrated for the dual case in Figure 7.2. Let $\{u_b^{\mathbf{A}}(t)\}_{b\in(\mathcal{B},*)}$ and $\{u_a^{\mathbf{B}}(t)\}_{a\in(\mathcal{A},\bullet)}$ be the eigenfunctions of $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$, respectively (recall that $(\tilde{\mathcal{A}},\tilde{\bullet}) = (\mathcal{B},*)$ and $(\tilde{\mathcal{B}},\tilde{*}) = (\mathcal{A},\bullet)$). Furthermore let (t_0,f_0) be the intersection of the TF loci of $u_{\beta_0}^{\mathbf{A}}(t)$ and $u_{\alpha_0}^{\mathbf{B}}(t)$, with α_0 and β_0 the group identity elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. Now $\mathbf{A}_\alpha$ or,

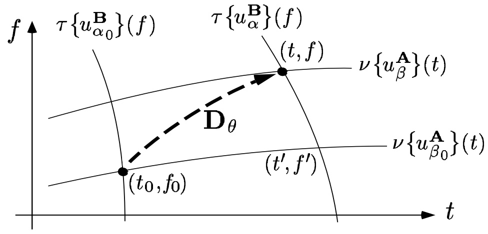

FIGURE 7.2. *Equivalence of localization function and displacement function in the dual case.*

equivalently, $e_{\mathbf{A}}(\cdot,\alpha)$ maps (t_0,f_0) to the intersection (t',f') of the TF loci of $u^{\mathbf{A}}_{\beta_0}(t)$ and (see (7.7.56), cf. Figure 7.1) $u^{\tilde{\mathbf{A}}}_{\alpha_0\bullet\alpha}(t) = u^{\mathbf{B}}_{\alpha_0\bullet\alpha}(t) = u^{\mathbf{B}}_{\alpha}(t)$, where we have used that $\mathbf{B}_\beta$ is the dual operator $\tilde{\mathbf{A}}_\beta$ of $\mathbf{A}_\alpha$ and assumed appropriate parameterization of $u^{\mathbf{B}}_{\alpha}(t)$. Subsequently, $\mathbf{B}_\beta$ or, equivalently, $e_{\mathbf{B}}(\cdot,\beta)$ maps (t',f') to the intersection (t,f) of the TF loci of $u^{\mathbf{B}}_{\alpha}(t)$ and $u^{\mathbf{A}}_{\beta_0*\beta}(t) = u^{\mathbf{A}}_{\beta}(t)$. Thus, for given (fixed) (t_0,f_0), the composite extended DF $e\big((t_0,f_0),\,(\alpha,\beta)\big) = e_{\mathbf{B}}\big(e_{\mathbf{A}}((t_0,f_0),\alpha),\beta\big) = e_{\mathbf{B}}((t',f'),\beta) = (t,f)$ maps the parameters (α,β) of the eigenfunctions $u^{\mathbf{B}}_{\alpha}(t)$ and $u^{\mathbf{A}}_{\beta}(t)$ to the intersection point (t,f) of their TF loci (see Figure 7.2). But this is exactly how we defined the LF $l(\alpha,\beta)$ in Subsection 7.9.2. Hence, the DF $d(\alpha,\beta) = e\big((t_0,f_0),\,(\alpha,\beta)\big)$ (with the specific (t_0,f_0) defined above) and the LF $l(a,b)$ are equivalent:

$$d(\alpha,\beta) = l(\alpha,\beta)\,. \tag{7.9.86}$$

For example, consider the case where $\mathbf{A}_\alpha$ is a warping operator and $\mathbf{B}_\beta$ is the modulation operator that is dual to $\mathbf{A}_\alpha$. In this case, the DF is $d(\alpha,\beta) = \big(m^{-1}(\alpha),\,\psi_{\mathcal{B}}(\beta)\,\tilde{m}'(m^{-1}(\alpha))\big)$ (see (7.7.65)) and the LF is $l(a,b) = \big(m^{-1}(a),\,\psi_{\mathcal{B}}(b)\,\tilde{m}'(m^{-1}(a))\big)$ (see (7.9.80) with $\tilde{n}(t) = \tilde{m}(t)$ and $\tilde{\mathcal{A}} = \mathcal{B}$ due to duality). Thus, LF and DF are recognized to be equivalent.

With this equivalence of DF and LF in the dual case, it immediately follows from Theorem 7.28 that the TF representations corresponding to $P_{x,y}(a,b)$ and $B_{x,y}(\alpha,\beta)$ are identical:

Corollary 7.29 *[26, 28, 30] Let $\{\mathbf{B}_\beta\}_{\beta\in(\mathcal{B},*)}$ be dual to $\{\mathbf{A}_\alpha\}_{\alpha\in(\mathcal{A},\bullet)}$. Let $\big\{\tilde{P}_{x,y}(t,f)\big\}$ denote the class of TF distributions based on $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$, as given by (7.9.78), (7.9.74) with $\mathbf{Q}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$. Let $\big\{\tilde{B}_{x,y}(t,f)\big\}$ be the class of covariant BTFRs based on the DO $\mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta\mathbf{A}_\alpha$, as given by (7.8.72). Then these classes are identical, i.e.,*

$$\big\{\tilde{P}_{x,y}(t,f)\big\} = \big\{\tilde{B}_{x,y}(t,f)\big\}\,.$$

Moreover, specific BTFRs of either class are identical, i.e.,

$$\tilde{P}_{x,y}(t,f) \;\equiv\; \tilde{B}_{x,y}(t,f)\,,$$

if the operator $\mathbf{K}$ *defining* $\tilde{B}_{x,y}(t,f)$ *is related to the kernel* $\phi(\alpha,\beta)$ *defining* $\tilde{P}_{x,y}(t,f)$ *according to (7.9.84).*

Proof. Using in turn (7.9.78), (7.9.86), (7.9.83), and (7.8.73), we have

$$\begin{aligned}\tilde{P}_{x,y}(t,f) &= P_{x,y}(a,b)\big|_{(a,b)=l^{-1}(t,f)} = P_{x,y}(a,b)\big|_{(a,b)=d^{-1}(t,f)} \\ &= B_{x,y}(a,b)\big|_{(a,b)=d^{-1}(t,f)} = \tilde{B}_{x,y}(t,f),\end{aligned}$$

where due to Theorem 7.28, $\mathbf{K}$ and $\phi(\alpha,\beta)$ are related according to (7.9.84). The equivalence of the entire classes $\{\tilde{P}_{x,y}(t,f)\}$ and $\{\tilde{B}_{x,y}(t,f)\}$ then follows from (7.9.82). □

7.10 Extension to General LCA Groups

From our assumption of continuous-time, nonperiodic signals and corresponding assumptions on the DF, it followed that the groups $(\mathcal{A},\bullet)$ and $(\mathcal{B},*)$ underlying the partial DOs $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ are isomorphic to $(\mathbb{R},+)$ (see Theorem 7.18 and Corollaries 7.19 and 7.20). Such groups belong to the larger class of *locally compact abelian (LCA)* groups [36, 37]. For LCA groups one can define a *dual* group, which is unique up to isomorphisms [36, 37]. For instance, the group $(\mathbb{R},+)$ is self-dual, and the group given by the interval $[0,1)$ with addition modulo 1 is dual to the group of integers with addition. For an LCA group $(\mathcal{G},\star)$ that is isomorphic to $(\mathbb{R},+)$, the dual group $(\tilde{\mathcal{G}},\tilde{\star})$ is also isomorphic to $(\mathbb{R},+)$. Conversely, any two groups that are isomorphic to $(\mathbb{R},+)$ are dual LCA groups.

Many results of this chapter can be extended to general LCA groups. This enables the construction of, e.g., discrete-time, periodic-frequency signal representations [49, 61, 62]. In particular, the definition of modulation operators (Definition 7.1) can be extended to arbitrary LCA groups simply by replacing the complex exponential $e^{j2\pi\psi_{\mathcal{H}}(h)\tilde{m}(t)}$ in (7.2.14) with $\gamma(h,m(t))$, where $\gamma(h,g)$ is a continuous *character* [36, 37] of the LCA group $(\mathcal{H},\diamond)$. Similarly, the warping operators (Definition 7.2) can be extended to LCA groups if the normalizing factor $\sqrt{|w_g'(t)|}$ in (7.2.18) is replaced by a suitable function (since differentiation may not be defined). Theorem 7.3 and Corollary 7.4 hold—with the exception of differentiability—for general LCA groups $(\mathcal{G},\star)$ and $(\mathcal{H},\diamond)$, provided they are *dual* groups [49].

The concept of dual operators as discussed in Section 7.3 can be generalized in a very natural way to *dual* LCA groups; again, the complex exponential $e^{j2\pi\,\psi_{\mathcal{G}}(g)\,\psi_{\mathcal{H}}(h)}$ in, e.g., (7.3.23) has to be replaced with the character $\gamma(g,h)$ of the dual LCA groups $(\mathcal{G},\star)$ and $(\mathcal{H},\diamond)$. Theorem 7.6 holds, except that $(\mathcal{G},\star)\times(\mathcal{H},\diamond)$ is not always isomorphic to $(\mathbb{R}^2,+)$. Theorem 7.7 holds if the frequency-shift and time-shift operators are replaced by the group diagonal and group translation operators, respectively [29, 34, 49].

The concept of affine operators (Section 7.4) can also be generalized to arbitrary LCA groups $(\mathcal{G}, \star), (\mathcal{H}, \diamond)$ if $\mu(g, h)$ in (7.4.29) is replaced by a more general function [49]. Theorem 7.9 essentially holds, with $\mathbf{D}_{g,h}$ being a unitary representation of a *semidirect product* [38] of $(\mathcal{G}, \star)$ and $(\mathcal{H}, \diamond)$. It seems that Theorem 7.10 *cannot* be generalized, since the scaling operator has no analogue in the general LCA setting.

All results of Section 7.5 can immediately be generalized to arbitrary topological groups; no further assumptions about the group are used.

In Section 7.6, we showed that under our assumptions on the TF point set $\mathcal{Z}$ and the DF, the underlying group must be isomorphic to either $(\mathbb{R}^2, +)$ or the affine group. Hence, under these assumptions, the results of Sections 7.6–7.8 cannot be extended to more general groups. However, for other sets $\mathcal{Z}$ (e.g., a grid in the TF plane), groups that are not isomorphic to $(\mathbb{R}^2, +)$ or to the affine group are appropriate [49, 62]. The extension of the DF concept to such groups is an interesting topic for future research.

The characteristic function method of Section 7.9 was generalized to LCA groups in [34]. Again, the key is to replace complex exponentials by group characters, and also Fourier transforms by *group Fourier transforms* [36, 37]. The equivalence of the covariance method and characteristic function method in the group domain (Theorem 7.28) holds also for dual operators based on dual LCA groups [49]. It is unclear whether the equivalence of these methods in the TF domain (Corollary 7.29) can be likewise extended since this would require an extension of the DF to general LCA groups.

7.11 Conclusion

The covariance theory allows to construct the class of linear or quadratic/bilinear time-frequency representations that are covariant to a given displacement operator, and thus it provides a theoretical basis for time-frequency analysis. Important classes of time-frequency representations—such as the short-time Fourier transform, the wavelet transform, Cohen's class, and the affine class—can all be derived from this principle.

The present development of the covariance method assumed that the groups underlying the partial displacement operators are isomorphic to $(\mathbb{R}, +)$. However, many of the results can be generalized to arbitrary LCA groups [49], thus allowing for discrete-time and/or periodic signals.

Apart from the covariance method as such, we discussed several concepts that are interesting in a broader context, namely, modulation and warping operators, dual and affine pairs of operators, and a general method for constructing the displacement function that describes the action of a displacement operator in the time-frequency plane. We also showed that for dual operators the covariance method and characteristic function method—two methods based on totally different principles—are equivalent.

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7.12 References

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